

Deck 03 · Transforms, forward kinematics, and the Jacobian

2.3 Homogeneous transforms and forward kinematics

2.4 The Jacobian and differential inverse kinematics

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Given every encoder reading, where is the hand?

Which way does it move when each motor turns a little, and where does that stop working?

The change-of-frame equation, and two extensions

$${}^A\mathbf{p}^P = {}^A\mathbf{p}^B + {}^A\mathbf{R}^B {}^B\mathbf{p}^P$$

1. **Chains.** Seven frames in a row $\rightarrow$ one matrix product: the **homogeneous transform**. With joint angles inside: **forward kinematics** $\varphi(\mathbf{q})$.
2. **Derivatives.** Which way the hand moves per joint: the **Jacobian** $\mathbf{J}(\mathbf{q}) = \partial\varphi/\partial\mathbf{q}$.

Why a 4×4 matrix

$${}^W\mathbf{p}^E = {}^W\mathbf{p}^1 + {}^W\mathbf{R}^1({}^1\mathbf{p}^2 + {}^1\mathbf{R}^2({}^2\mathbf{p}^3 + \dots))$$

Correct, but unreadable, and it grows one nesting level per joint.

The trick: append a 1 to every position and put rotation and translation into one matrix.

The homogeneous transform

$${}^A\mathbf{X}^B = \begin{bmatrix} {}^A\mathbf{R}^B & {}^A\mathbf{p}^B \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \text{SE}(3), \quad {}^A\tilde{\mathbf{p}}^P = {}^A\mathbf{X}^B {}^B\tilde{\mathbf{p}}^P$$
$$\begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} {}^B\mathbf{p}^P \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} {}^B\mathbf{p}^P + \mathbf{p} \\ 1 \end{bmatrix}$$

The first three rows are the change-of-frame equation of 2.2; the fourth says $1 = 1$.

Composition: inner letters match

$${}^A\mathbf{X}^C = {}^A\mathbf{X}^B {}^B\mathbf{X}^C$$

Blocks: ${}^A\mathbf{R}^C = {}^A\mathbf{R}^B {}^B\mathbf{R}^C$ and
 ${}^A\mathbf{p}^C = {}^A\mathbf{p}^B + {}^A\mathbf{R}^B {}^B\mathbf{p}^C$.

RAI prints it as $T_{W \rightarrow C} =$
 $T_{W \rightarrow A} T_{A \rightarrow B} T_{B \rightarrow C}$; we write the
 monogram form.

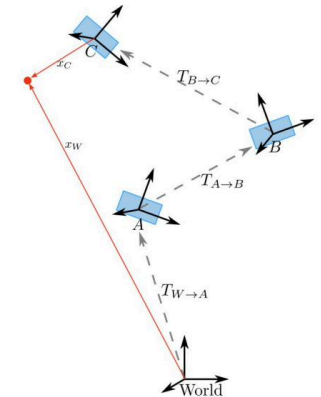
Composition of transforms

$X \sim T$: same

here:

$${}^W X^A \equiv T_{W \rightarrow A}$$

$$x_W \equiv {}^W p_W^{\text{red_point}}$$



$$T_{W \rightarrow C} \equiv T_{W \rightarrow A} T_{A \rightarrow B} T_{B \rightarrow C}$$

$$x_W = T_{W \rightarrow A} T_{A \rightarrow B} T_{B \rightarrow C} x_C$$

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Invert a pose: subtract the offset, then rotate back

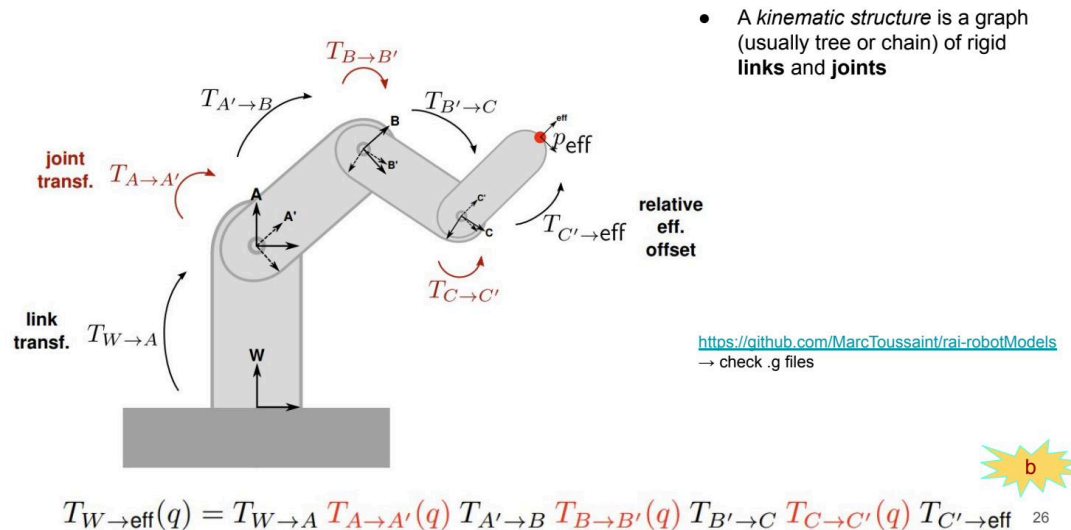
$${}^B\mathbf{p}^P = \mathbf{R}^\top ({}^A\mathbf{p}^P - \mathbf{p})$$

$$({}^A\mathbf{X}^B)^{-1} = {}^B\mathbf{X}^A = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

Transpose the **rotation**; rotate and negate the **translation**.

For a desired pose ${}^A\mathbf{X}^{B_d}$, the relative pose error is $({}^A\mathbf{X}^B)^{-1} {}^A\mathbf{X}^{B_d} = {}^B\mathbf{X}^{B_d}$: the desired frame seen from the current one, identity when the poses agree.

A robot is a kinematic tree



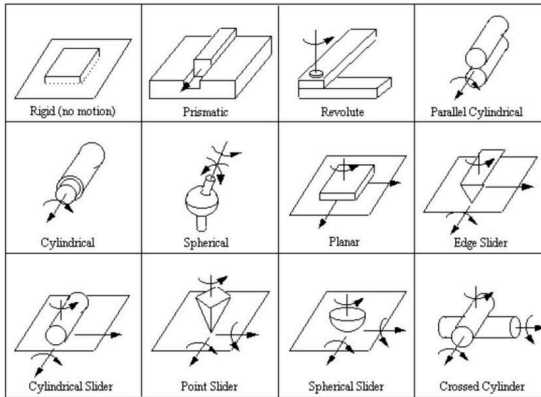
- **Links:** constant transforms, from the geometry
- **Joints:** a hinge or a slider, one number q_i each; revolute turns, prismatic slides
- Walk from the world to the end-effector and multiply, one factor per edge:

$${}^W \mathbf{X}^E(\mathbf{q}) = \mathbf{X}_0 \mathbf{X}_1(q_1) \mathbf{X}_2(q_2) \cdots \mathbf{X}_n(q_n) \mathbf{X}_{n+1}$$

Forward kinematics $\varphi(\mathbf{q})$: constant factors from the geometry, one moving factor per joint.

Joint types, and the tree as a .g file

Joint types



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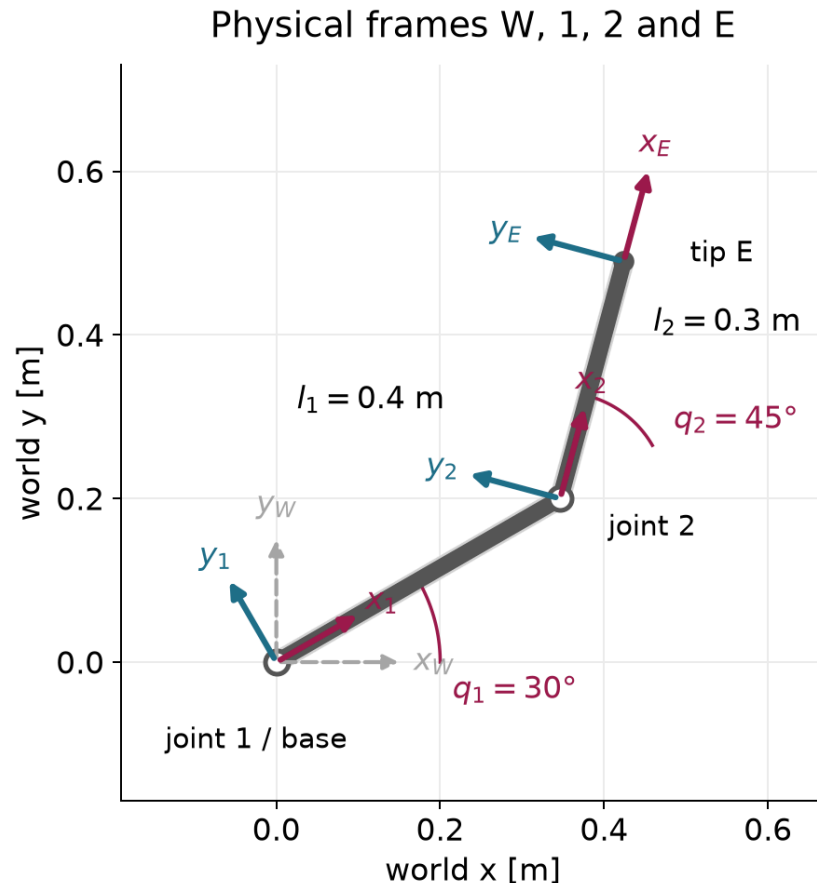
our arms use only revolute and prismatic joints; the other types are combinations of the two

$$\mathbf{X}_{\text{rev}}(q_i) = \begin{bmatrix} \mathbf{R}_z(q_i) & 0 \\ \mathbf{0}^\top & 1 \end{bmatrix}, \quad \mathbf{X}_{\text{pri}}(q_i) = \begin{bmatrix} \mathbf{I} & (q_i, 0, 0)^\top \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

```
joint1 (world): {  
  joint: hingeZ, q: [0.5236]  
}  
joint2 (joint1): {  
  Q: [0.40 0 0],  
  joint: hingeZ, q: [0.7854]  
}  
endeff (joint2): {  
  Q: [0.30 0 0], shape: marker  
}
```

The tree, written down: parent in parentheses, constant offsets as `Q:`, one number per `joint:`.

The two-link planar arm



$$\mathbf{q} = (30^\circ, 45^\circ), \quad l_1 = 0.4 \text{ m}, \quad l_2 = 0.3 \text{ m}$$

Frame 1 sits at the base and points along link 1. Frame 2 sits at the elbow and points along link 2. Frame E sits at the tip.

The elbow angle is relative to link 1, so the tip's heading is

$$q_1 + q_2 = 75^\circ.$$

We will compute the pose of E in W for **this one fixed configuration**.

Four factors describe one fixed configuration

The first turn aligns the axes with link 1.

$$\mathbf{R}(30^\circ) \approx \begin{bmatrix} .8660 & -.5000 & 0 \\ .5000 & .8660 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second turn aligns the axes with link 2.

$$\mathbf{R}(45^\circ) \approx \begin{bmatrix} .7071 & -.7071 & 0 \\ .7071 & .7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first shift reaches the elbow in those axes.

$$\mathbf{T}(.4, 0) = \begin{bmatrix} 1 & 0 & .4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second shift reaches the tip in those axes.

$$\mathbf{T}(.3, 0) = \begin{bmatrix} 1 & 0 & .3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^W\mathbf{X}^E = \underbrace{\mathbf{R}(q_1)}_{{}^W\mathbf{X}^1} \underbrace{\mathbf{T}(l_1, 0)}_{{}^1\mathbf{X}^2} \underbrace{\mathbf{R}(q_2)\mathbf{T}(l_2, 0)}_{{}^2\mathbf{X}^E}$$

Here $\mathbf{R}(\theta)$ and $\mathbf{T}(x, y)$ are homogeneous 3×3 factors. The rotation block is $\mathbf{R}_2(\theta)$.

Appending a factor rotates its shift

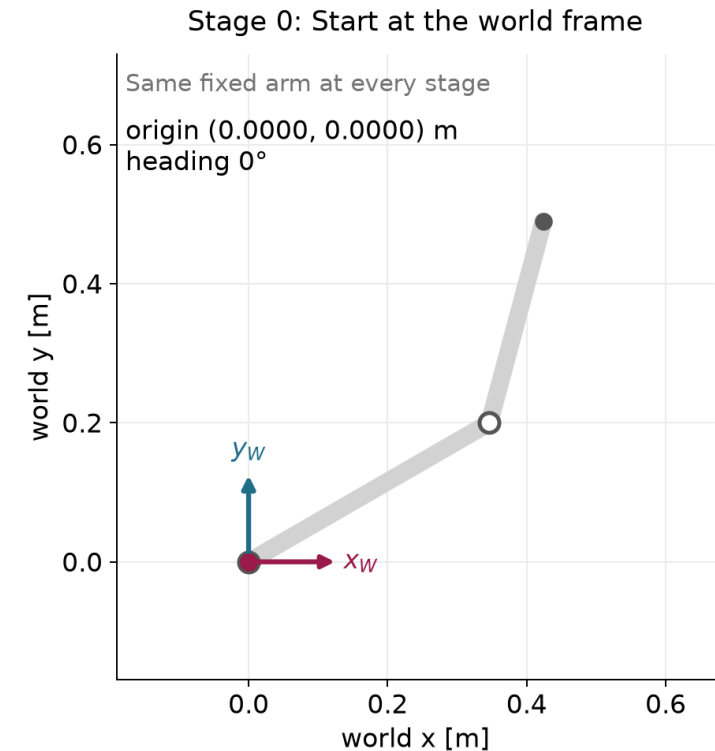
Multiplication updates the axes and the origin separately.

$$\begin{bmatrix} \mathbf{R}_{\text{old}} & \mathbf{p}_{\text{old}} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\text{next}} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{\text{old}}\mathbf{R}_{\text{next}} & \mathbf{p}_{\text{old}} + \mathbf{R}_{\text{old}}\mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

The next shift is in the **current frame's axes**, so rotate it before adding it to the world position.

$$\mathbf{X}^{(0)} = \mathbf{I}_3, \quad \mathbf{X}^{(s+1)} = \mathbf{X}^{(s)} \text{ (next factor)}$$

The superscript s counts construction stages. It is neither a link number nor time.

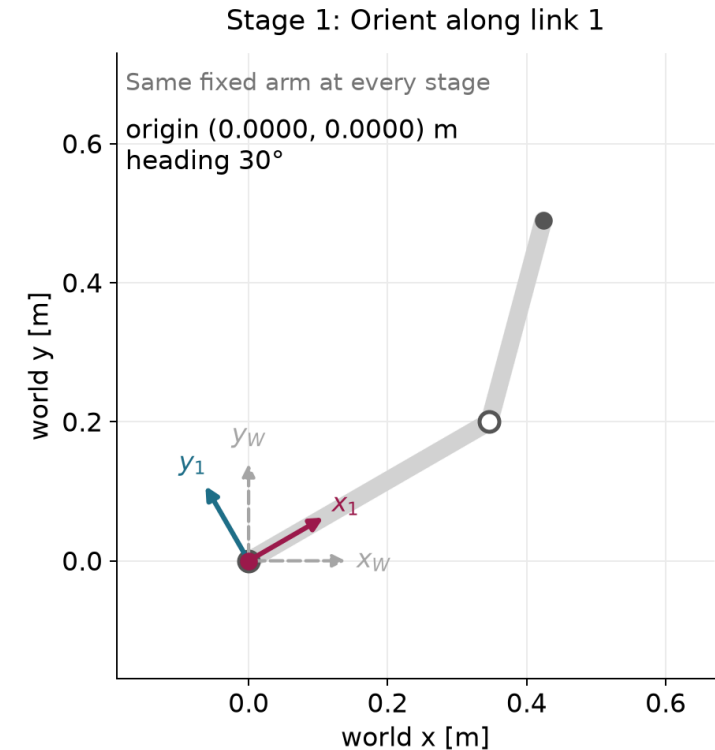


Turn the frame at joint 1

The first joint sets the direction of link 1. Its origin stays at the base.

$$\mathbf{X}^{(1)} = {}^W\mathbf{X}^1 = \mathbf{I}_3 \mathbf{R}(30^\circ) \approx \begin{bmatrix} .8660 & -.5000 & 0 \\ .5000 & .8660 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{p}_{\text{new}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathbf{I}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first column points at 30° .



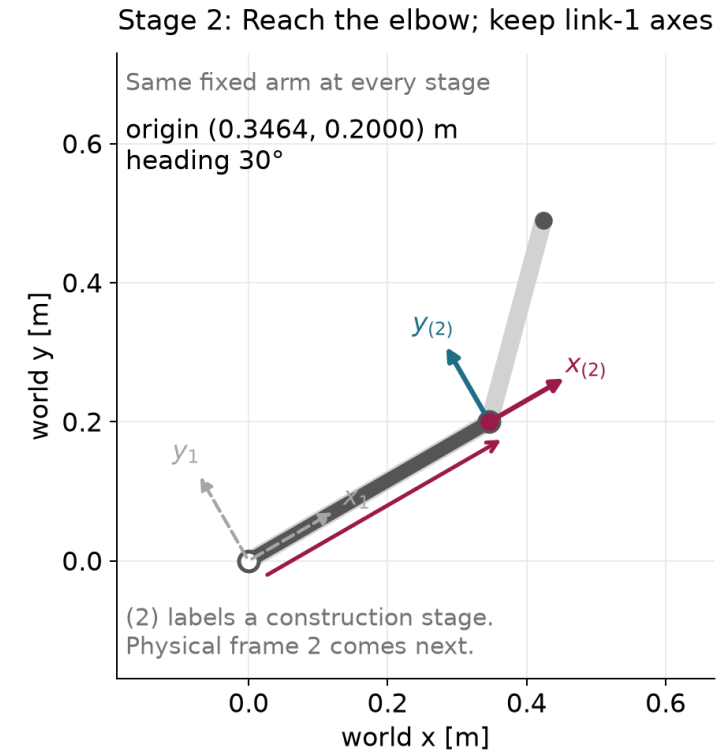
Shift to joint 2 along the turned x-axis

The elbow is 0.4 m along link 1, so its displacement must be rotated into world axes.

$$\mathbf{p}_{\text{new}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} .4 \\ 0 \end{bmatrix} \approx \begin{bmatrix} .3464 \\ .2000 \end{bmatrix}$$

$$\mathbf{X}^{(2)} = \mathbf{X}^{(1)} \mathbf{T}(.4, 0) \approx \begin{bmatrix} .8660 & -.5000 & .3464 \\ .5000 & .8660 & .2000 \\ 0 & 0 & 1 \end{bmatrix}$$

The axes still point at 30° . This intermediate construction frame is at the elbow; physical frame 2 also includes the next turn.



Turn the frame at joint 2

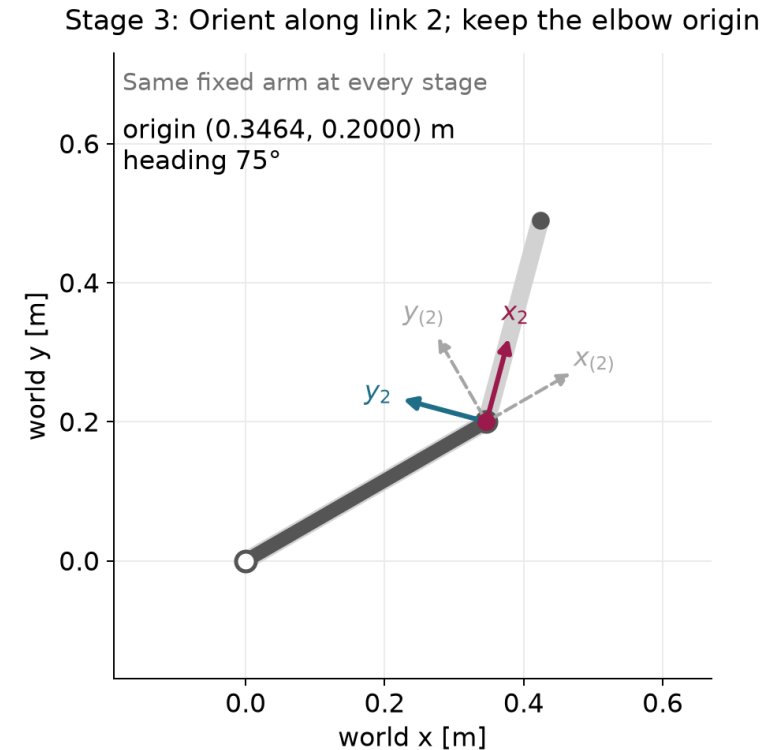
The elbow turn adds 45° to the heading. Its zero translation leaves the elbow origin fixed.

$$\mathbf{X}^{(3)} = \mathbf{X}^{(2)} \mathbf{R}(45^\circ) = {}^W\mathbf{X}^2 \approx \begin{bmatrix} .2588 & -.9659 & .3464 \\ .9659 & .2588 & .2000 \\ 0 & 0 & 1 \end{bmatrix}$$

Each new rotation entry is a row times a column. For the new x-axis:

$$\begin{aligned} \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ &\approx .2588, \\ \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ &\approx .9659. \end{aligned}$$

These are the world components of the 75° x-axis.



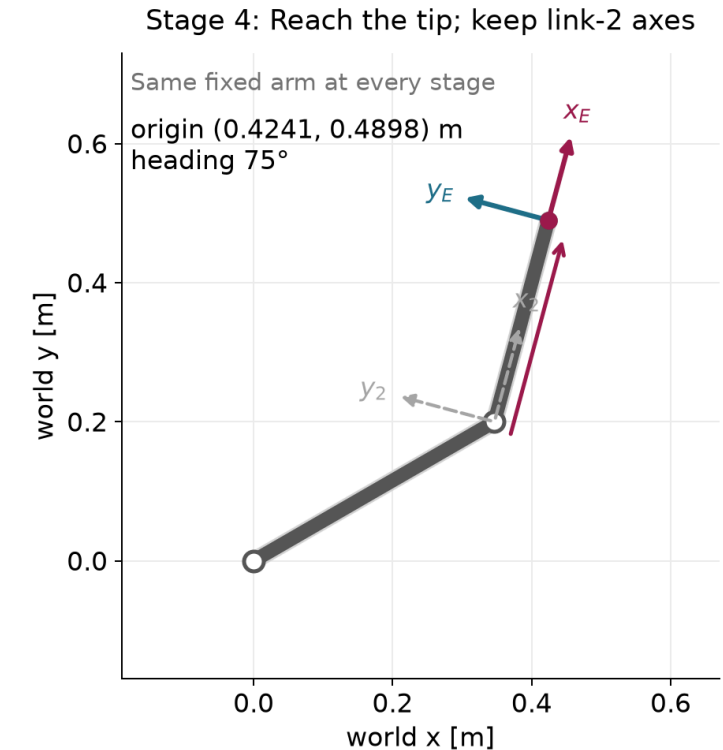
Shift to the tip along its x-axis

The tip is 0.3 m along the new x-axis. Add that displacement to the elbow position.

$$\mathbf{p}_{\text{new}} = \begin{bmatrix} .4 \cos 30^\circ \\ .4 \sin 30^\circ \end{bmatrix} + \mathbf{R}_2(75^\circ) \begin{bmatrix} .3 \\ 0 \end{bmatrix}$$
$$\mathbf{p}_{\text{new}} \approx \begin{bmatrix} .34641 + .07765 \\ .20000 + .28978 \end{bmatrix} \approx \begin{bmatrix} .4241 \\ .4898 \end{bmatrix}$$

$$\mathbf{X}^{(4)} = \mathbf{X}^{(3)} \mathbf{T}(.3, 0) = {}^W \mathbf{X}^E \approx \begin{bmatrix} .2588 & -.9659 & .4241 \\ .9659 & .2588 & .4898 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation preserves the 75° heading.



Multiply the full pose chain in one shot

The four stages are one matrix product. The displayed entries are rounded.

$$\begin{aligned}
 {}^W\mathbf{X}^E &\approx \underbrace{\begin{bmatrix} .8660 & -.5000 & 0 \\ .5000 & .8660 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}(30^\circ)} \underbrace{\begin{bmatrix} 1 & 0 & .4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}(.4,0)} \underbrace{\begin{bmatrix} .7071 & -.7071 & 0 \\ .7071 & .7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}(45^\circ)} \underbrace{\begin{bmatrix} 1 & 0 & .3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}(.3,0)} \\
 &\approx \begin{bmatrix} \textcolor{red}{.2588} & \textcolor{blue}{-.9659} & .4241 \\ \textcolor{red}{.9659} & \textcolor{blue}{.2588} & .4898 \\ 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

The coloured columns give the tip's axes in W . Its position column, in metres, gives the task we will differentiate:

$$\varphi(\mathbf{q}) = {}^W\mathbf{p}^E(\mathbf{q}) = \begin{bmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{bmatrix}.$$

Apply the same product to coordinates from the right

The tip's origin has coordinates zero in E . Each factor rewrites the **same point** in the preceding axes.

$${}^W\tilde{\mathbf{p}}^E = \mathbf{R}(30^\circ)\mathbf{T}(.4, 0)\mathbf{R}(45^\circ)\mathbf{T}(.3, 0) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Apply this factor next	The point's homogeneous coordinates	These coordinates use
Start at the right	$(0, 0, 1)^\top$	tip frame E
$\mathbf{T}(.3, 0)$	$(.3, 0, 1)^\top$	physical frame 2
$\mathbf{R}(45^\circ)$	$\approx (.2121, .2121, 1)^\top$	stage-2 elbow axes
$\mathbf{T}(.4, 0)$	$\approx (.6121, .2121, 1)^\top$	physical frame 1
$\mathbf{R}(30^\circ)$	$\approx (.4241, .4898, 1)^\top$	world frame W

Lift the planar pose into 3D

We insert a physical z-coordinate and keep the homogeneous 1 last.

$$\underbrace{\begin{bmatrix} .2588 & -.9659 & .4241 \\ .9659 & .2588 & .4898 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{planar pose, rounded}} \longrightarrow \underbrace{\begin{bmatrix} .2588 & -.9659 & 0 & .4241 \\ .9659 & .2588 & 0 & .4898 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{same pose in 3D, rounded}}$$

$$(x, y, 1)^\top \longrightarrow (x, y, 0, 1)^\top$$

The third rotation column is the unchanged z-axis. The third position entry is zero. A spatial arm uses these same 4×4 products with axes that point in different directions.

The FR3 uses the same transform chain

The model supplies each fixed link offset and joint axis. Joint readings supply the seven angles. We use this supplied configuration.

$$\mathbf{q} = (0, -\pi/4, 0, -3\pi/4, 0, \pi/2, \pi/4).$$

Each joint combines its fixed placement with a turn about its local z-axis:

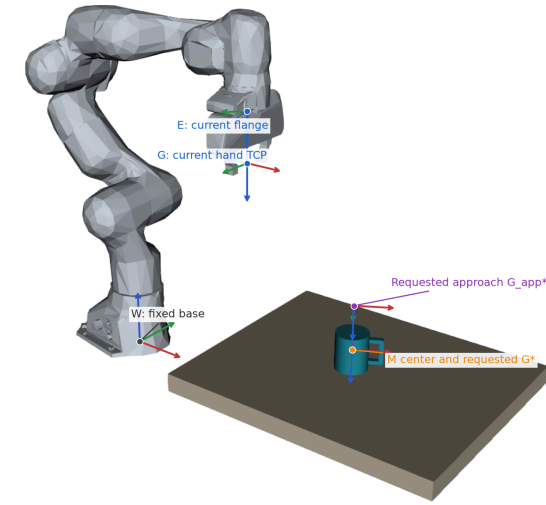
$${}^{i-1}\mathbf{X}^i(q_i) = \begin{bmatrix} \mathbf{R}_{\text{origin},i} & \mathbf{p}_{\text{origin},i} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_z(q_i) & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}.$$

Here the physical FR3 links are numbered 0 through 7.

The fixed base is W ; E is the flange; G is the hand's tool center point.

FR3 model at the exact ready configuration

Current pose comes from the joints. The mug pose and grasp choice supply the targets.



G now: (0.3069, 0.0000, 0.4869) m G*: (0.5000, 0.1000, 0.1000) m G_app*: (0.5000, 0.1000, 0.2000) m

Axes: x red, y green, z blue. Official collision shapes are used as low-detail visuals. No IK or contact is solved.

The FR3 scene uses the manufacturer's frame chain. The mug pose is supplied.

Build one FR3 joint transform from the model

Joint 2 has zero fixed translation. Its fixed rotation is about x, and its variable rotation is about local z.

$${}^1\mathbf{X}^2(-\pi/4) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{fixed } \mathbf{R}_x(-\pi/2)} \underbrace{\begin{bmatrix} c & c & 0 & 0 \\ -c & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{joint } \mathbf{R}_z(-\pi/4)} = \begin{bmatrix} c & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & -c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad c = \frac{\sqrt{2}}{2}.$$

Joint 1 has zero angle, so its pose is a pure 0.333 m rise from the base.

$${}^W\mathbf{X}^2 = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & .333 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^W\mathbf{X}^1} \underbrace{\begin{bmatrix} c & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & -c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^1\mathbf{X}^2} = \begin{bmatrix} c & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & -c & 0 & .333 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Carry a link offset into world axes

Link 3 lies 0.316 m along link 2's negative y-axis. Rotate this local offset:

$${}^W\mathbf{p}^3 = {}^W\mathbf{p}^2 + {}^W\mathbf{R}^2 \begin{bmatrix} 0 \\ -0.316 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.333 \end{bmatrix} + \begin{bmatrix} -0.316/\sqrt{2} \\ 0 \\ 0.316/\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} -0.2234 \\ 0 \\ 0.5564 \end{bmatrix} \text{ m.}$$

Repeat for all seven joints, then append the fixed flange transform:

$${}^W\mathbf{X}^E(\mathbf{q}) = {}^W\mathbf{X}^0 \left(\prod_{i=1}^7 {}^{i-1}\mathbf{X}^i(q_i) \right) {}^7\mathbf{X}^E \approx \begin{bmatrix} .7071 & -.7071 & 0 & .3069 \\ -.7071 & -.7071 & 0 & 0 \\ 0 & 0 & -1 & .5903 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the flange to the grasp point

The manufacturer places the hand's tool center point G between the fingers. Its offset from E is fixed.

$${}^E\mathbf{X}^G \approx \begin{bmatrix} .7071 & .7071 & 0 & 0 \\ -.7071 & .7071 & 0 & 0 \\ 0 & 0 & 1 & .1034 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^W\mathbf{X}^G(\mathbf{q}) = {}^W\mathbf{X}^E(\mathbf{q}){}^E\mathbf{X}^G \approx \begin{bmatrix} 1 & 0 & 0 & .3069 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & .4869 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{q} = (0, -\pi/4, 0, -3\pi/4, 0, \pi/2, \pi/4).$$

This rotates the hand axes by -45° about the flange z-axis and shifts 0.1034 m along that axis.

$${}^W\mathbf{p}^G = {}^W\mathbf{p}^E + {}^W\mathbf{R}^E \begin{bmatrix} 0 \\ 0 \\ .1034 \end{bmatrix} \approx \begin{bmatrix} .3069 \\ 0 \\ .5903 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -.1034 \end{bmatrix} = \begin{bmatrix} .3069 \\ 0 \\ .4869 \end{bmatrix} \text{ m.}$$

From the mug to the desired grasp

The supplied mug frame M is at its center, with a 30° yaw.

$${}^W\mathbf{X}^M \approx \begin{bmatrix} .8660 & -.5000 & 0 & .50 \\ .5000 & .8660 & 0 & .10 \\ 0 & 0 & 1 & .10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Choose the desired tool frame G^* at that center, with its x-axis along the mug handle and its z-axis down.

$${}^M\mathbf{X}^{G^*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matching inner letters gives the requested pose in world coordinates.

$${}^W\mathbf{X}^{G^*} = {}^W\mathbf{X}^M {}^M\mathbf{X}^{G^*} \approx \begin{bmatrix} .8660 & .5000 & 0 & .50 \\ .5000 & -.8660 & 0 & .10 \\ 0 & 0 & -1 & .10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The chosen grasp flips the mug's y- and z-axes. Its zero offset keeps the mug-center position.

Current G comes from the joints. Desired G^* comes from the mug and the grasp choice.

Choose an approach pose before asking for joints

1. Choose the offset. Stand off by 0.10 m along negative grasp z. That axis points down, so the approach is above the mug.

2. Compose the poses. Append the local translation to the desired grasp pose.

$$\underbrace{\begin{bmatrix} .8660 & .5000 & 0 & .50 \\ .5000 & -.8660 & 0 & .10 \\ 0 & 0 & -1 & .10 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{W\mathbf{X}^{G^*}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -.10 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{G^*\mathbf{X}^{G_{\text{app}}^*}} \approx \underbrace{\begin{bmatrix} .8660 & .5000 & 0 & .50 \\ .5000 & -.8660 & 0 & .10 \\ 0 & 0 & -1 & .20 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{W\mathbf{X}^{G_{\text{app}}^*}}$$

3. Ask for the joints. Inverse kinematics (IK) seeks joint values that put the actual tool at this approach pose.

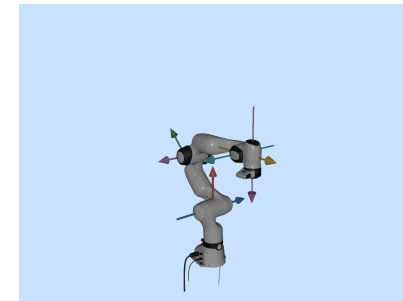
$$W\mathbf{X}^G(\mathbf{q}) = W\mathbf{X}^{G_{\text{app}}^*}.$$

Live demo: the Panda's 7-joint arm in Drake

```
fk_jacobian_demo.py --meshcat
```

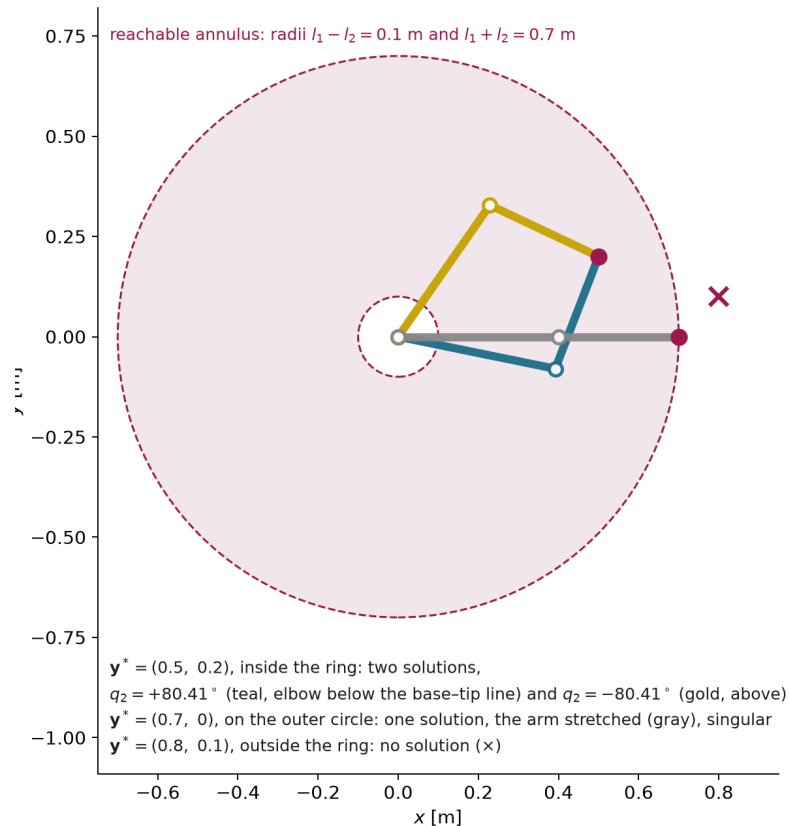
- Seven joint sliders; the flange follows: **forward kinematics, live**.
The coloured arrows show the positive joint axes
- Predict: joint 1 turns in the positive direction from the ready pose. Which way does the hand go?
- Watch ${}^W\mathbf{p}^E$ change; the console's rank and $\sigma_{\min}$ come back on the Panda singularity slide

In the stack: item 1 of the map. The **Goal box** asks it to reach a target; **Reference motion** uses it for collisions.



ready pose, positive joint-axis arrows (--png)

Inverse kinematics: from a target to joint positions



Given a desired tip position or pose, find joint positions. This is **inverse kinematics (IK)**.

For the two-link position task, seek

$$\varphi(\mathbf{q}) = \mathbf{y}^*.$$

- Two-link position task: **none** outside the ring; **one** on a boundary; **two** inside
- Seven joints, full pose fixed: often a **continuum** of postures

Our two-link arm admits an **analytic formula**.

We focus on **numerical IK**: one formulation for different arms, posture preferences and constraints.

Deck 04: derive error-reducing steps using FK and its Jacobian.

Deck 05: add joint limits and clearance constraints.

Summary of 2.3

- A pose is one 4×4 matrix; the extra 1 turns "rotate and add" into one product
- Compose by multiplying with matching inner letters; invert with $\mathbf{R}^\top$ and $-\mathbf{R}^\top \mathbf{p}$
- The chain product gives the tip pose. A position-only task keeps its position coordinates.

Question	Given	Compute or seek
Forward kinematics	Joint positions	Tip position or pose
Inverse kinematics	Desired tip position or pose	Joint positions

Our two-link FK example used joint angles $(30^\circ, 45^\circ)$ to compute tip position $(0.4241, 0.4898)$ m and heading 75° .

Next, we predict **tip velocity from joint velocities**. That local motion map will also help us improve an IK guess.

Break

Second half: which way does the tip move when a joint turns, and where does that fail?

The Jacobian: a table of small joint effects

Forward kinematics tells us where the tip is. To move it, we also need to predict **how joint velocities make the tip move**.

$$\Delta \mathbf{x} \approx \mathbf{J}(\mathbf{q}) \Delta \mathbf{q}, \quad J_{ij} = \frac{\partial x_i}{\partial q_j}$$

- **Column j :** what moving joint j does to the task
- **Row i :** how all joints contribute to task component i
- m task components, n joints: $\mathbf{J} \in \mathbb{R}^{m \times n}$

Divide by a short time interval and take its limit. This gives **differential forward kinematics**:

$$\underbrace{\dot{\mathbf{x}}}_{\text{task velocity}} = \mathbf{J}(\mathbf{q}) \underbrace{\dot{\mathbf{q}}}_{\text{joint velocities}}$$

$\mathbf{x}$ is the **task-coordinate vector**, such as the tip's position coordinates. Full pose motion uses the six-row geometric Jacobian introduced on the 3D slide.

The two-link Jacobian, derived

1. Name each entry. Rows are tip coordinates; columns are joints.

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{bmatrix}$$

2. Differentiate the position.

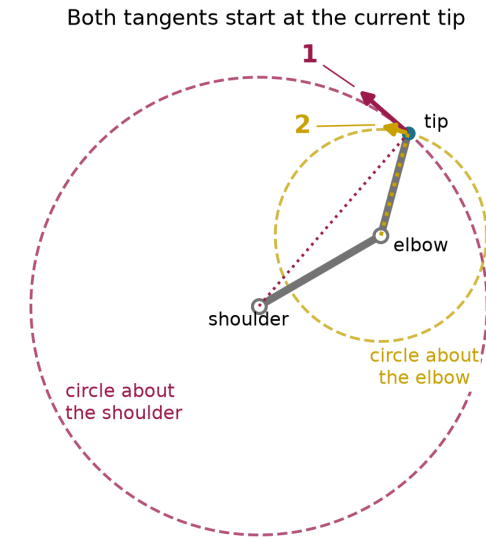
$$x = l_1 c_1 + l_2 c_{12},$$

$$y = l_1 s_1 + l_2 s_{12}.$$

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$c_1 = \cos q_1, \quad s_1 = \sin q_1,$$

$$c_{12} = \cos(q_1 + q_2), \quad s_{12} = \sin(q_1 + q_2).$$



- **Joint 1 alone:** the tip moves tangent to the shoulder's circle.
- **Joint 2 alone:** the tip moves tangent to the elbow's circle.

Each column gives tip motion **per radian** of that joint.

Read the numbers as joint effects

1. Substitute the worked pose and link lengths.

$$l_1 = 0.4 \text{ m}, \quad l_2 = 0.3 \text{ m}, \quad q_1 = 30^\circ, \quad q_1 + q_2 = 75^\circ.$$

$$\mathbf{J}_{\text{pos}} = \begin{bmatrix} -0.4 \sin 30^\circ - 0.3 \sin 75^\circ & -0.3 \sin 75^\circ \\ 0.4 \cos 30^\circ + 0.3 \cos 75^\circ & 0.3 \cos 75^\circ \end{bmatrix}$$

2. Evaluate the entries.

$$\mathbf{J}_{\text{pos}} \approx \begin{bmatrix} -0.4898 & -0.2898 \\ 0.4241 & 0.0776 \end{bmatrix} \text{ m/rad}$$

3. Use the same matrix for displacement or velocity.

Joint displacement input (rad):

$$\Delta \mathbf{p} \approx \mathbf{J}_{\text{pos}} \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \approx \begin{bmatrix} -0.004898 \\ 0.004241 \end{bmatrix} \text{ m}$$

About **4.90 mm left, 4.24 mm up**.

Joint velocity input (rad/s):

$$\dot{\mathbf{p}} = \mathbf{J}_{\text{pos}} \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \approx \begin{bmatrix} -0.004898 \\ 0.004241 \end{bmatrix} \text{ m/s}$$

Differential FK: about 4.90 mm/s left, 4.24 mm/s up.

First order only

turn joint 1 by	actual tip motion	$\mathbf{J} \Delta \mathbf{q}$	error
5°	$(-0.0443, 0.0351) \text{ m}$	$(-0.0427, 0.0370) \text{ m}$	2.5 mm
30°	$(-0.3017, 0.1464) \text{ m}$	$(-0.2564, 0.2220) \text{ m}$	9 cm

The Jacobian is a **local** statement about φ at one $\mathbf{q}$.

Add heading: the same arm, one more task row

1. Describe the planar pose. Add the gripper's heading to its position coordinates.

$$(\theta, x, y)^\top, \quad \theta = q_1 + q_2.$$

These are one angle and two position coordinates, not a point in 3D.

2. Differentiate the heading.

$$\begin{aligned} \frac{\partial \theta}{\partial q_1} &= 1, & \frac{\partial \theta}{\partial q_2} &= 1. \\ \dot{\theta} &= \frac{\partial \theta}{\partial q_1} \dot{q}_1 + \frac{\partial \theta}{\partial q_2} \dot{q}_2 \\ &= \dot{q}_1 + \dot{q}_2. \end{aligned}$$

3. Add the heading row above the position rows.

$$\mathbf{J}_{\text{full}} = \begin{bmatrix} \frac{\partial \theta}{\partial q_1} & \frac{\partial \theta}{\partial q_2} \\ \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{bmatrix}$$

4. Compute the gripper motion at this pose.

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} \approx \begin{bmatrix} 1 & 1 \\ -0.4898 & -0.2898 \\ 0.4241 & 0.0776 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}.$$

The inputs $\dot{q}_1$ and $\dot{q}_2$ are **joint angular velocities** (rad/s). The outputs are gripper angular velocity (rad/s) and tip velocity (m/s).

In 3D: three turning rates, three moving rates

Reaching the right point is not enough for a grasp. The fingers must also face the right way. Angular velocity describes how we turn them toward that orientation.

ω is the **angular-velocity vector**, expressed in world axes.

- Its **direction** gives the rotation axis (right-hand rule).
- Its **magnitude** gives angular speed (rad/s).

For the planar arm, rotation is about the world z-axis:

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}.$$

A zero vector means no instantaneous rotation.

$\mathbf{v}$ is the **linear velocity of E 's origin**, in m/s (the flange in the Panda demos).

Differential forward kinematics: at the current joint positions, multiply the full Jacobian by joint angular velocities to compute the flange's turning and moving velocities.

Spatial velocity (world axes, angular first):

$${}^W\mathbf{V}^E = \begin{bmatrix} \omega \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{\text{ang}} \\ \mathbf{J}_{\text{pos}} \end{bmatrix} \dot{\mathbf{q}}$$

Task	Panda Jacobian
Position only: v_x, v_y, v_z	3×7
Full motion: $\omega_x, \omega_y, \omega_z, v_x, v_y, v_z$	6×7

Build the angular Jacobian from the joint axes

We reuse the **FR3 ready pose**. Each hinge rotates about its own **local z-axis**.

1. Read both rotations at the ready pose. Joint 1 has zero angle and no fixed rotation.

$${}^W\mathbf{R}^1 = \mathbf{I}_3, \quad {}^W\mathbf{R}^2 = \begin{bmatrix} c & c & 0 \\ 0 & 0 & 1 \\ c & -c & 0 \end{bmatrix}.$$

Here $\mathbf{I}_3$ is the identity rotation, and $c = \sqrt{2}/2$.

2. Convert each unit joint axis to world coordinates.

$${}^W\hat{\mathbf{z}}_1 = {}^W\mathbf{R}^1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$${}^W\hat{\mathbf{z}}_2 = {}^W\mathbf{R}^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

3. Stack the directions and multiply.

The first two columns of $\mathbf{J}_{\text{ang}}$ are

$$[{}^W\hat{\mathbf{z}}_1 \quad {}^W\hat{\mathbf{z}}_2] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Move joints 1 and 2 at **0.1 and 0.2 rad/s**. Hold the others still.

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix} \text{ rad/s}.$$

Multiply each column by its joint velocity, then add.

Differential inverse kinematics, the square case

We computed gripper motion from joint velocities. **Now choose the gripper motion and find the joint velocities.** We use the **two-link position task** for the calculation.

To lift a mug, request an **upward tip velocity**:

$$\dot{\mathbf{p}}_{\text{cmd}} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ m/s.}$$

The current joint positions determine $\mathbf{J}_{\text{pos}}$.
Solve for the unknown joint velocities:

$$\mathbf{J}_{\text{pos}}(\mathbf{q}) \underbrace{\dot{\mathbf{q}}}_{\text{find}} = \underbrace{\dot{\mathbf{p}}_{\text{cmd}}}_{\text{requested}}.$$

This is **differential inverse kinematics**. When the Jacobian is square and invertible,

$$\dot{\mathbf{q}} = \mathbf{J}_{\text{pos}}^{-1} \dot{\mathbf{p}}_{\text{cmd}}.$$

At the worked joint positions,

$$\mathbf{q} = \begin{bmatrix} 30^\circ \\ 45^\circ \end{bmatrix}, \quad \mathbf{J}_{\text{pos}} \approx \begin{bmatrix} -0.4898 & -0.2898 \\ 0.4241 & 0.0776 \end{bmatrix} \text{ m/rad.}$$

The solution is

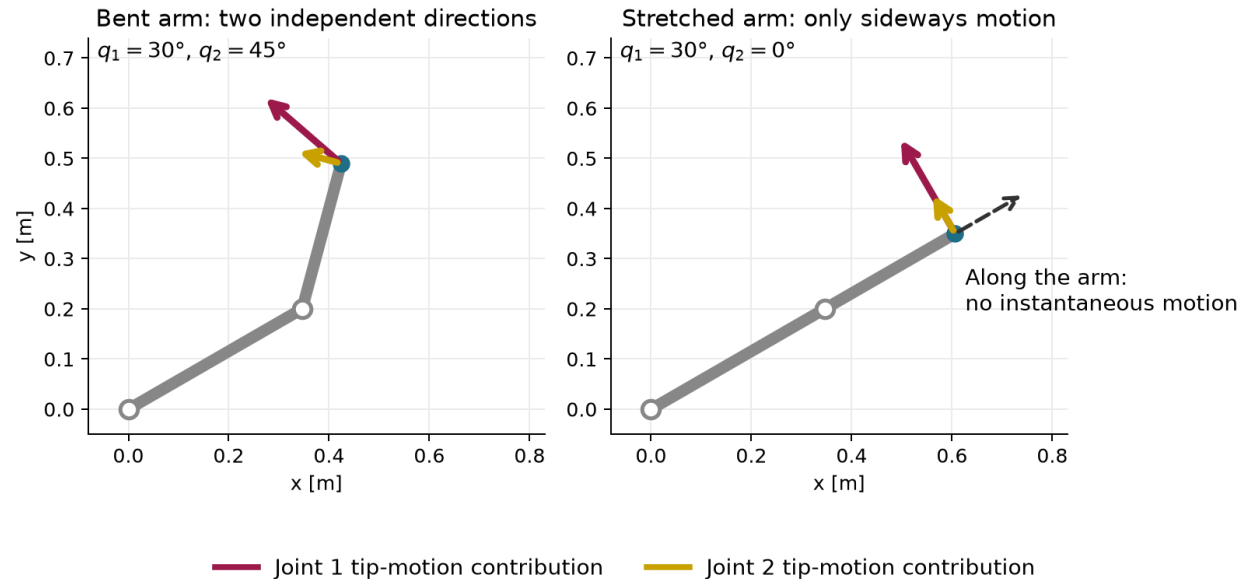
$$\dot{\mathbf{q}} \approx \begin{bmatrix} 0.3415 \\ -0.5772 \end{bmatrix} \text{ rad/s.}$$

Check the achieved tip velocity:

$$\mathbf{J}_{\text{pos}} \dot{\mathbf{q}} \approx \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ m/s.}$$

At these joint positions, specifying both tip-velocity components determines a **unique pair of joint angular velocities**.

Why the inverse can fail



Open the two-link module. Compare **Worked example** with **Stretched**.

At the stretched pose, both joint effects point **sideways**. No combination produces instantaneous tip motion **along the arm**. Sideways motion remains possible.

Rank counts independent tip-motion directions:

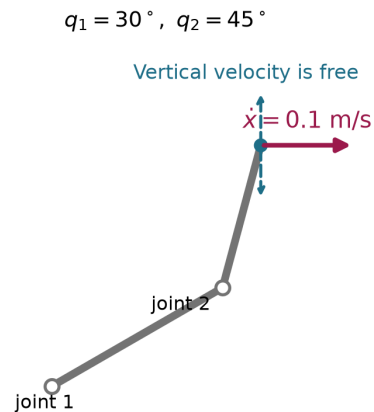
$$\text{rank}(\mathbf{J}_{\text{pos}}) : 2 \longrightarrow 1.$$

A drop below the task's maximum attainable rank is a **singularity**.

Near this pose, some tip requests need **large joint velocities**. Optimization I introduces damping to reduce them.

Several joint motions can satisfy one request

Previously, we requested both tip-velocity components. Now request **only horizontal velocity**; vertical velocity is free.



$$\mathbf{J}_{\text{pos}} \approx \begin{bmatrix} -0.4898 & -0.2898 \\ 0.4241 & 0.0776 \end{bmatrix} \text{ m/rad.}$$

Its first row is $\mathbf{J}_x$, used on the right.

Keep $q_1 = 30^\circ$ and $q_2 = 45^\circ$. Use the horizontal row:

$$\underbrace{\begin{bmatrix} -0.4898 & -0.2898 \end{bmatrix}}_{\mathbf{J}_x} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = 0.1 \text{ m/s.}$$

Both choices satisfy the request:

Choice	$\dot{q}_1$ (rad/s)	$\dot{q}_2$ (rad/s)	$\dot{x}$ (m/s)
A: joint 1 only	-0.2042	0	0.1000
B: joint 2 only	0	-0.3451	0.1000

One requested component, two joint velocities: there are more answers to choose from.

The pseudo-inverse selects one of those answers

Given a requested task velocity, the **pseudo-inverse** is the matrix that selects the joint velocities with the closest achievable match and the smallest norm among ties.

For a wide Jacobian with independent rows (full row rank),

$$\mathbf{J}^+ = \mathbf{J}^\top (\mathbf{J}\mathbf{J}^\top)^{-1}, \quad \dot{\mathbf{q}} = \mathbf{J}^+ \dot{\mathbf{x}}.$$

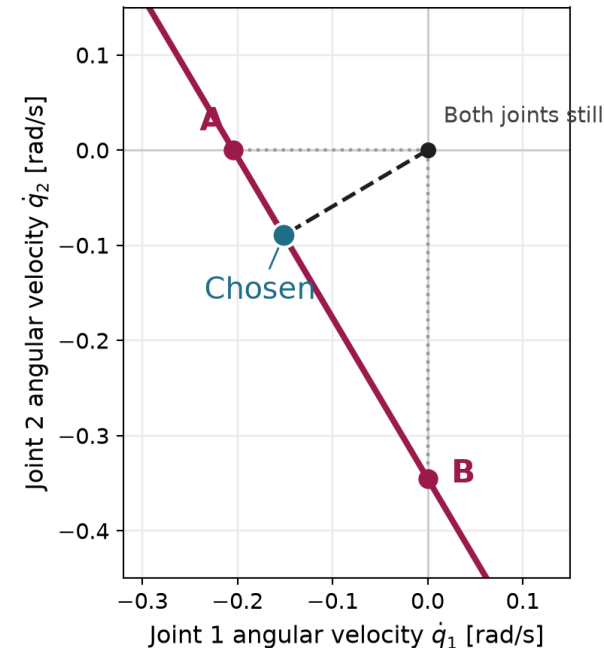
For this nonzero one-row Jacobian $\mathbf{J}\mathbf{J}^\top$ is the scalar sum of squared entries, so transpose the row and divide:

$$\mathbf{J}_x^+ = \frac{\begin{bmatrix} -0.4898 \\ -0.2898 \end{bmatrix}}{(-0.4898)^2 + (-0.2898)^2} \approx \begin{bmatrix} -1.5123 \\ -0.8948 \end{bmatrix} \text{ rad/m.}$$

Multiply by the requested horizontal speed:

$$\dot{\mathbf{q}} = \mathbf{J}_x^+ (0.1) \approx \begin{bmatrix} -0.1512 \\ -0.0895 \end{bmatrix} \text{ rad/s.}$$

This is the valid point **nearest the origin** in the graph.



One equation in two unknowns leaves a one-dimensional set; the equation is linear in $\dot{\mathbf{q}}$, so the set is a line, and moving along it changes only the vertical tip speed.

Extra joint motion can preserve the task

Start from A. Add coordinated joint motion to obtain B:

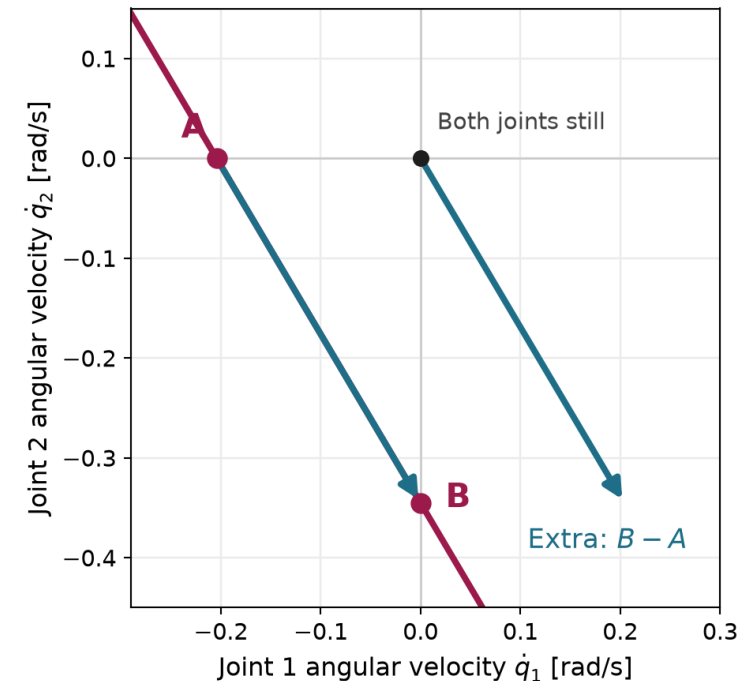
$$\underbrace{\begin{bmatrix} -0.2042 \\ 0 \end{bmatrix}}_A + \underbrace{\begin{bmatrix} 0.2042 \\ -0.3451 \end{bmatrix}}_{\text{extra joint velocities}} = \underbrace{\begin{bmatrix} 0 \\ -0.3451 \end{bmatrix}}_B \text{ rad/s.}$$

A and B give the same horizontal tip velocity. The extra motion therefore contributes **zero horizontal velocity**.

Joint velocities with zero effect on the chosen task form its **null space**:

$$\mathbf{J}_x \dot{\mathbf{q}}_{\text{null}} = 0.$$

Vertical velocity may still change.



The blue arrows show the same extra joint velocities:
adding them takes A to B.

The arm can change posture while the gripper stays fixed

The gripper is at the required grasp pose. Keep its **position and orientation**, while changing the arm's posture.

Several joints can move together so their effects at the gripper **cancel**.

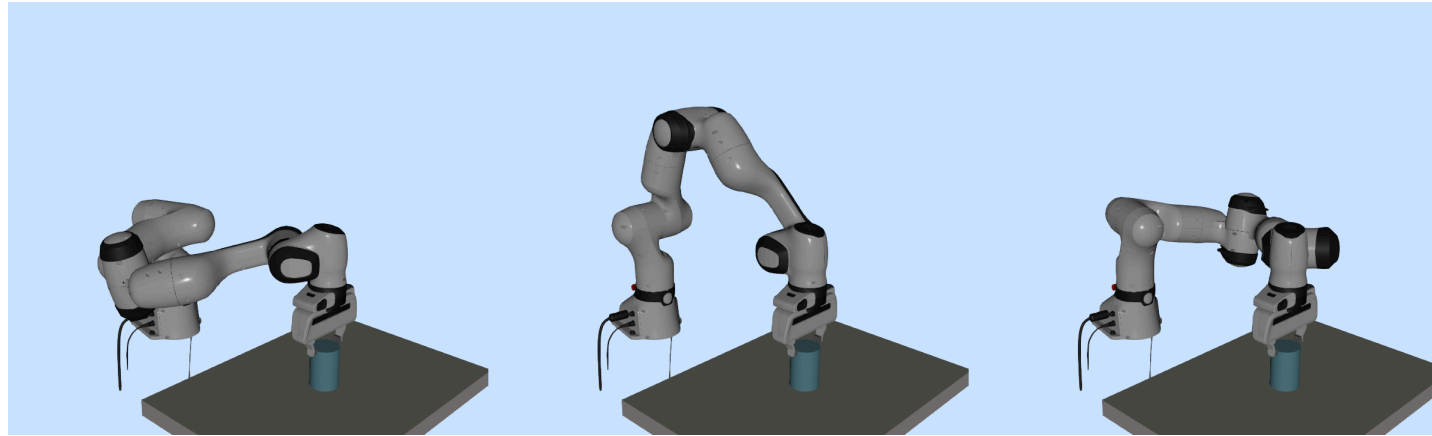
Now use the **full-pose geometric Jacobian**:

$$\mathbf{J} \dot{\mathbf{q}}_{\text{null}} = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{bmatrix} = \mathbf{0}.$$

Panda at a regular configuration	Count
Joint angular velocities	7
Independent gripper-motion components held at zero	6
Independent coordinated motion patterns remaining	1

That one remaining pattern moves several joints together while the gripper does not move. The next demo shows it: the elbow swings and the gripper pose stays fixed.

Live demo: redundancy, the elbow swings, the hand stays

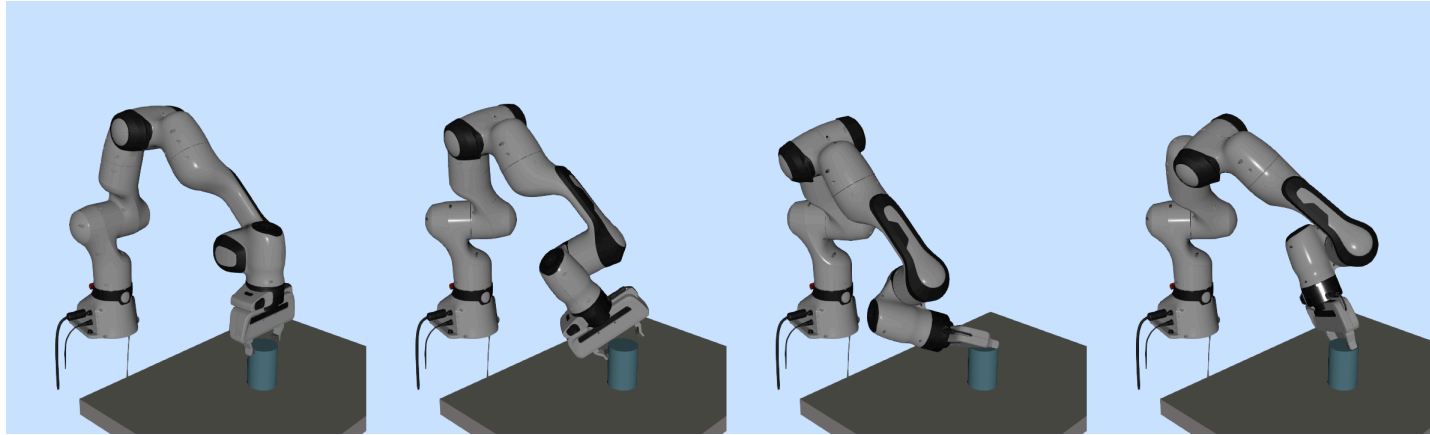


The flange pose stays fixed. The tool axis points down while the elbow rises and falls.

Watch the joints move **together**. The arm changes posture while the gripper keeps its position and orientation.

The demo repeats small null-space moves and corrects drift; the right panel's seven readouts show the joint angles in degrees.

Same demo, only the fingertip position fixed



Now hold the tool-centre point between the fingers. The gripper may turn while this point stays in place.

Relaxing the orientation requirement allows **more coordinated joint motions**.

At full position-task rank, seven joints and three position components leave **four independent motion patterns**.

The held point is now the tool center, 10.3 cm beyond the flange.

A velocity becomes a small position command

Back to the two-link arm of slide 36 and its upward tip command. The Panda demos showed what redundancy leaves free; these two slides turn a joint velocity, and then a target, into a joint change.

If the robot accepts joint positions, integrate over one short control period:

$$\mathbf{q}_{\text{next}} = \mathbf{q} + \Delta t \dot{\mathbf{q}}$$

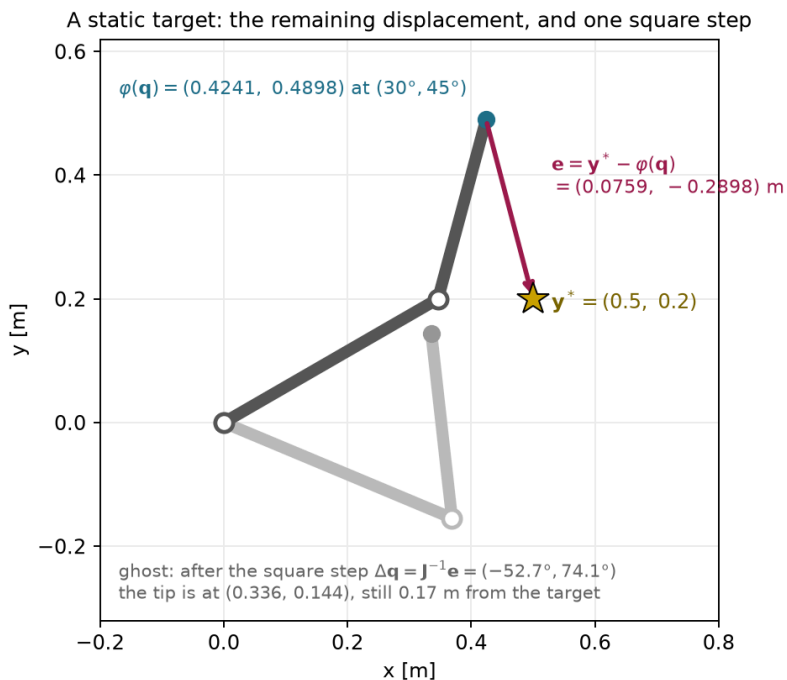
Upward tip command: $\dot{\mathbf{x}} = (0, 0.1)$ m/s.

At our pose $\dot{\mathbf{q}} = (0.3415, -0.5772)$ rad/s; with $\Delta t = 0.01$ s:

$$\Delta \mathbf{q} = (0.003415, -0.005772) \text{ rad}$$

About **1 mm upward**. Measure the new configuration and recompute.

From a target back to a joint change



quantity	value
target $\mathbf{y}^*$	$(0.5, 0.2) \text{ m}$
tip $\phi(\mathbf{q})$ at $(30^\circ, 45^\circ)$	$(0.4241, 0.4898) \text{ m}$
displacement $\mathbf{e} = \mathbf{y}^* - \phi(\mathbf{q})$, no time step	$(0.0759, -0.2898) \text{ m}$
local model	$\mathbf{J}(\mathbf{q}) \Delta \mathbf{q} \approx \mathbf{e}$
$\det \mathbf{J}$	0.0849: square, invertible
step $\Delta \mathbf{q} = \mathbf{J}^{-1} \mathbf{e}$	$(-52.7^\circ, 74.1^\circ)$, far past where the first-order model held

- **Own now:** the local problem and this one step
- **Deck 04 derives:** how far to trust it, the step for any $\mathbf{J}$, and "closest" when no $\mathbf{q}$ reaches $\mathbf{y}^*$

What an action asks the robot-side stack to do

Policy output	Typical conversion	Still robot-dependent
Joint targets or deltas	Track the targets; no IK conversion of this action	Joint meaning, limits, safety checks
End-effector pose	Form geometric error; solve IK	Reachability, posture, collisions
End-effector velocity	Differential IK; integrate if needed	Jacobian, rates and limits

“Delta” needs a frame, units and an update rule.

A common task-space interface can help transfer; it does not guarantee it.

What does learning replace, keep, and assume?

A1 is out; next: optimization

- **A1, the classical stack** (due Oct XX)
- **Deck 04 (3.1–3.2):** gradient descent, Newton, Gauss-Newton; the bridge slide's step derived, damped and repeated, on this arm
- **Backup:** motion gains, extra singularity examples and demo controls; algebra checks; damping, numerical IK and the null-space projection

Backup slides

Determinant algebra, inverse checks, the failure-mode recap, damping and numerical IK (both derived in Deck 04), and the optional null-space construction.

$$\det \mathbf{J} = l_1 l_2 \sin q_2$$

$$\det \mathbf{J} = l_1 l_2 \sin q_2, \quad \text{at } (30^\circ, 45^\circ) : 0.4 \cdot 0.3 \cdot 0.7071 = 0.0849$$

- Two lines of algebra from the two-link Jacobian derivation (notes §4.7); the l_2^2 terms cancel
- q_1 **is absent**: the base joint decides *where* the tip is, not *how well* it moves
- $\det \mathbf{J} = 0$ exactly when $\sin q_2 = 0$: arm stretched or folded (see the singular-configurations slide)

Backup: the inverse and a finite-difference check

For a square, nonsingular 2×2 Jacobian:

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

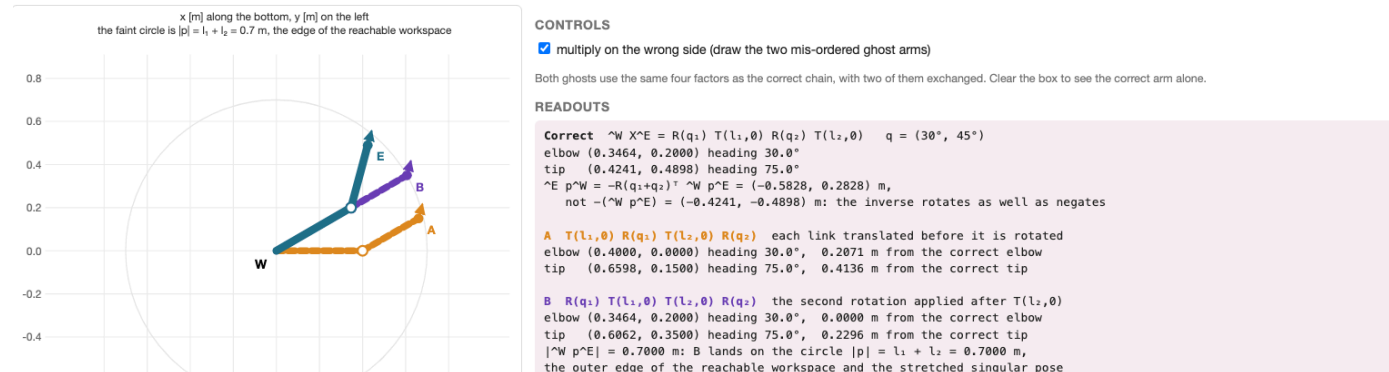
Check a column without differentiating by hand:

$$\mathbf{J}_{:,j} \approx \frac{\varphi(\mathbf{q} + h\mathbf{e}_j) - \varphi(\mathbf{q} - h\mathbf{e}_j)}{2h}$$

$\mathbf{e}_j$ selects joint j ; $h = 10^{-6}$ rad gives agreement to 3.7×10^{-11} here.

Failure modes of the day

- **Order of multiplication:** right acts in the body frame, left in the world frame
- **The missing 1:** $(p, 0)$ rotates but does not translate
- **Rows versus columns:** $\mathbf{J}$ is $m \times n$; $\mathbf{J}\dot{\mathbf{q}}$ needs an n -vector
- **"Singular means stuck";** and trusting $\det \mathbf{J}$ or the rank alone instead of $\sigma_{\min}$



modules/two-link-arm, wrong-order section: ghost A elbow 21 cm off and tip 41 cm off; ghost B elbow right and tip 23 cm off; heading right every time

Damping: accept error instead of excessive joint speed

A weak motion direction needs large joint rates to produce a small tip velocity.

Damped least squares balances two costs:

$$\min_{\dot{\mathbf{q}}} \underbrace{\|\mathbf{J}\dot{\mathbf{q}} - \dot{\mathbf{x}}\|^2}_{\text{tracking error}} + \rho \underbrace{\|\dot{\mathbf{q}}\|^2}_{\text{joint motion}}, \quad \rho > 0$$

Larger ρ : less joint motion, more accepted tracking error.

Damping regularises. Explicit constraints enforce speed, joint and collision limits.

Numerical IK: improve a guess, then linearise again

For a **position target**, the remaining displacement is

$$\mathbf{e} = \mathbf{y}^* - \varphi(\mathbf{q}).$$

$$\Delta \mathbf{q} = \mathbf{J}^+(\mathbf{q})\mathbf{e}, \quad \mathbf{q} \leftarrow \mathbf{q} + \Delta \mathbf{q}$$

Recompute the error and Jacobian; repeat. This is the Gauss–Newton step.

- A large step may need shortening or damping
- Unreachable targets and poor initial guesses can prevent convergence
- Full pose: a geometric rotation and position error in the Jacobian's frame; its residual derivative depends on the error convention; this block stays with position

The optimization block derives the step and adds constraints.

Backup: how to compute a null-space motion

Start with preferred joint velocities $\mathbf{z}$. They would cause task velocity $\mathbf{J}\mathbf{z}$.

The pseudo-inverse finds the smallest joint velocities that produce that same task velocity. Subtract them:

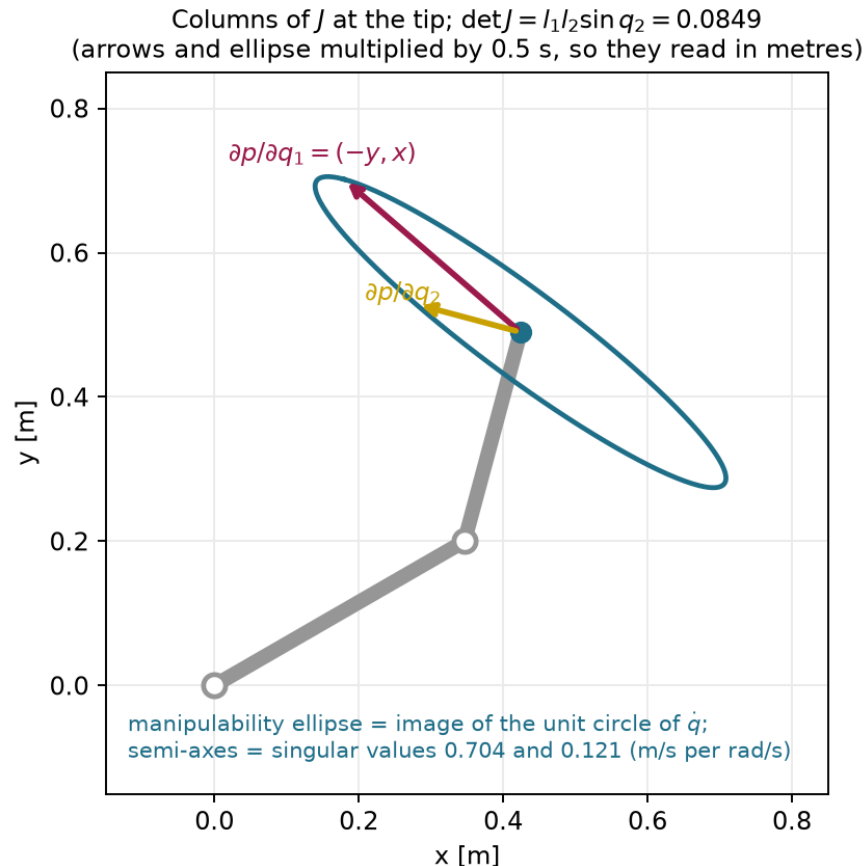
$$\dot{\mathbf{q}}_{\text{null}} = \mathbf{z} - \mathbf{J}^+ \mathbf{J}\mathbf{z}, \quad \mathbf{J}\dot{\mathbf{q}}_{\text{null}} = \mathbf{0}.$$

Add this zero-task motion to a primary velocity solution:

$$\dot{\mathbf{q}} = \mathbf{J}^+ \dot{\mathbf{x}} + \dot{\mathbf{q}}_{\text{null}}.$$

There are $n - \text{rank } \mathbf{J}$ independent null-space directions. This construction uses the plain pseudo-inverse at the current configuration.

Backup: motion gains, rank and singular values



We return to the **two-link position task** and its 2×2 **Jacobian J_{pos}** .

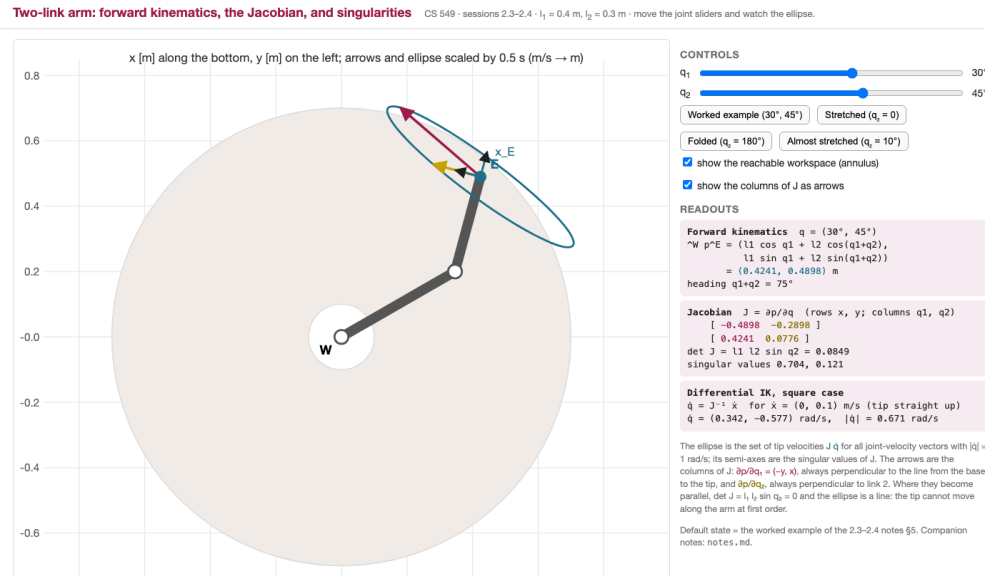
Rank of J_{pos} : number of independent tip-velocity directions the joints can produce. Here its rank is **2**.

The **ellipse boundary** traces tip velocities at unit joint-rate norm.

Its radii are the **singular values**: largest and smallest motion gains.

Here: $\sigma_1 = 0.7037$, $\sigma_2 = \sigma_{\min} = 0.1206$ m/rad. The short axis points almost along the arm.

Backup: explore the two-link module in more detail



<modules/two-link-arm/index.html>

- Start at the worked example
- $q_2 \rightarrow 0$: ellipse collapses, column arrows turn parallel
- Change q_1 : ellipse turns, shape unchanged
- Predict: upward 0.1 m/s at $q_2 = 1^\circ$ costs $2\times$, $10\times$ or $30\times$ the joint speed at 45° ? Then read $\dot{q}$ at $45^\circ, 10^\circ, 1^\circ$
- Press **Stretched**: rank drops from **2** to **1**. The tip can still move sideways.

Backup: the Panda's zero configuration is singular

Each turning joint adds angular velocity **along its own axis**.

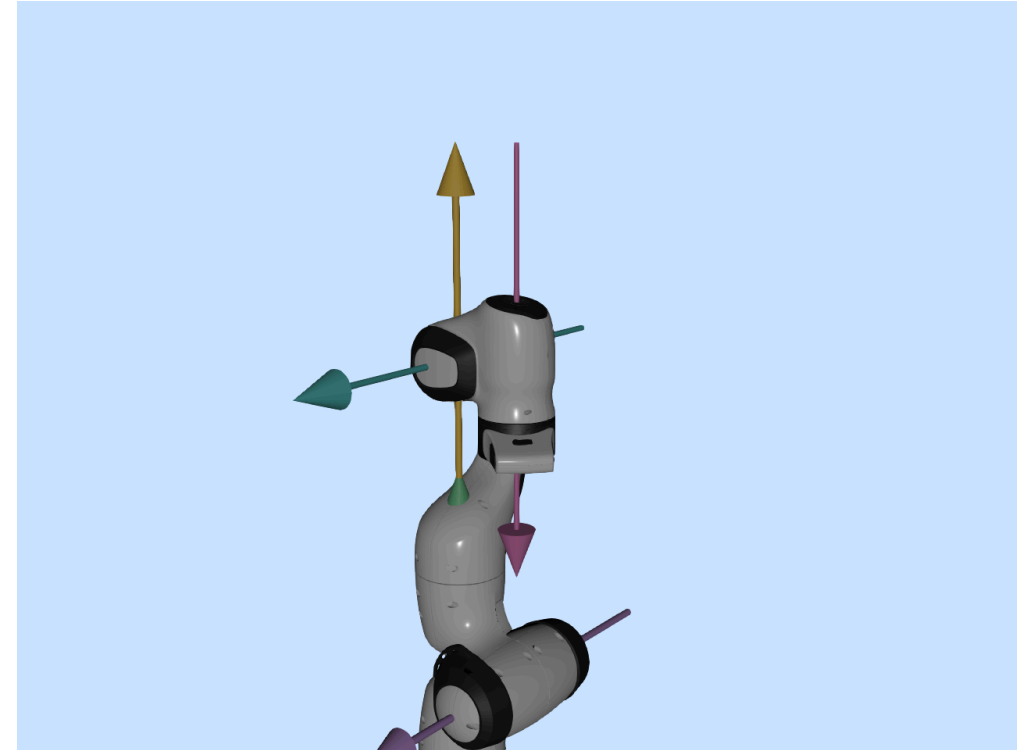
- Joints **1, 3, 5, 7** are parallel to world z .
- Joints **2, 4, 6** are parallel to world y .

Together, they cannot turn the flange **about world x** .

The full 6×7 geometric Jacobian loses one independent motion:

$$\text{rank}(\mathbf{J}) = 5, \quad \sigma_{\min} = 0.$$

The zero configuration is outside the Panda model's joint limits.



The arrows show the positive joint-axis directions near the wrist. [Open the demo](#)

Backup: large joint velocities near a singularity

Keep the **two-link position task**, world-up request and shoulder angle fixed:

$$\dot{\mathbf{p}}_{\text{cmd}} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ m/s}, \quad q_1 = 30^\circ.$$

q_2	$\det \mathbf{J}_{\text{pos}}$	$\dot{\mathbf{q}}$ (rad/s)	$\ \dot{\mathbf{q}}\ $ (rad/s)
45°	0.0849	(0.3415, -0.5772)	0.67
10°	0.0208	(0.9254, -1.8852)	2.10
1°	0.0021	(7.3778, -16.9275)	18.47

This request has a component in the arm's weakening motion direction, so the required joint speed grows as the arm straightens.

[Open the worked state](#). Set the elbow to 45° , 10° , 1° and read the differential-IK joint velocities.