Deck 07 · Statics and grasp selection

Contact forces, moments, and equilibrium

Grasp selection from observed surfaces

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We can plan a path to the box. Where should the fingers touch so they can hold it during the lift?

From reaching a pose to supporting a load

Reach a pose → **Support a load** → **Select a grasp**

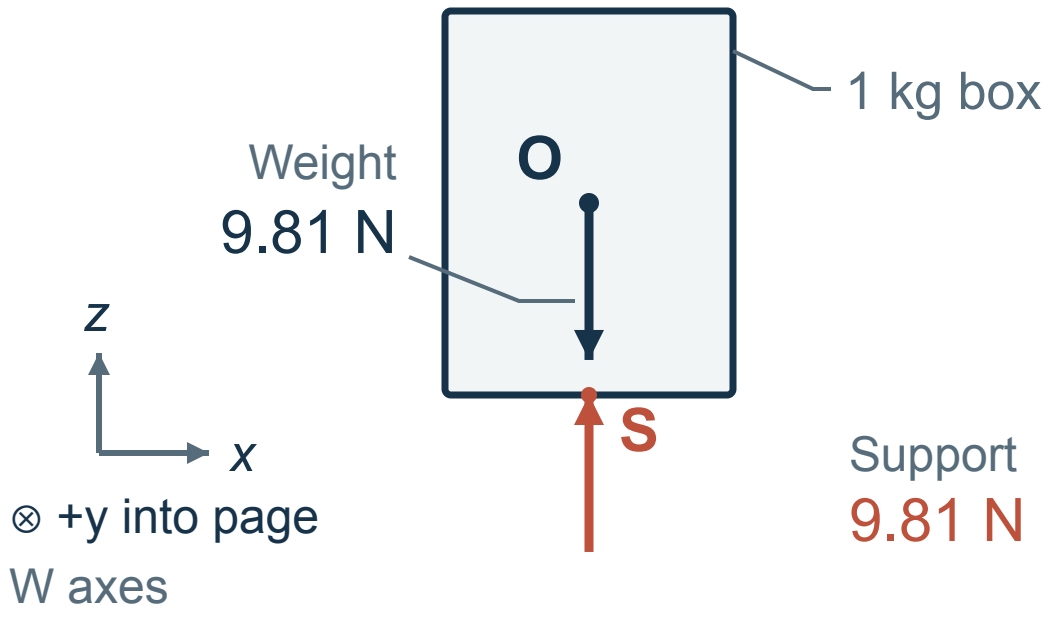
Frames, kinematics, planning

How contacts hold an object

Geometry and feasible loads

A reachable pose is only the starting point.

A force is a push or pull on the box

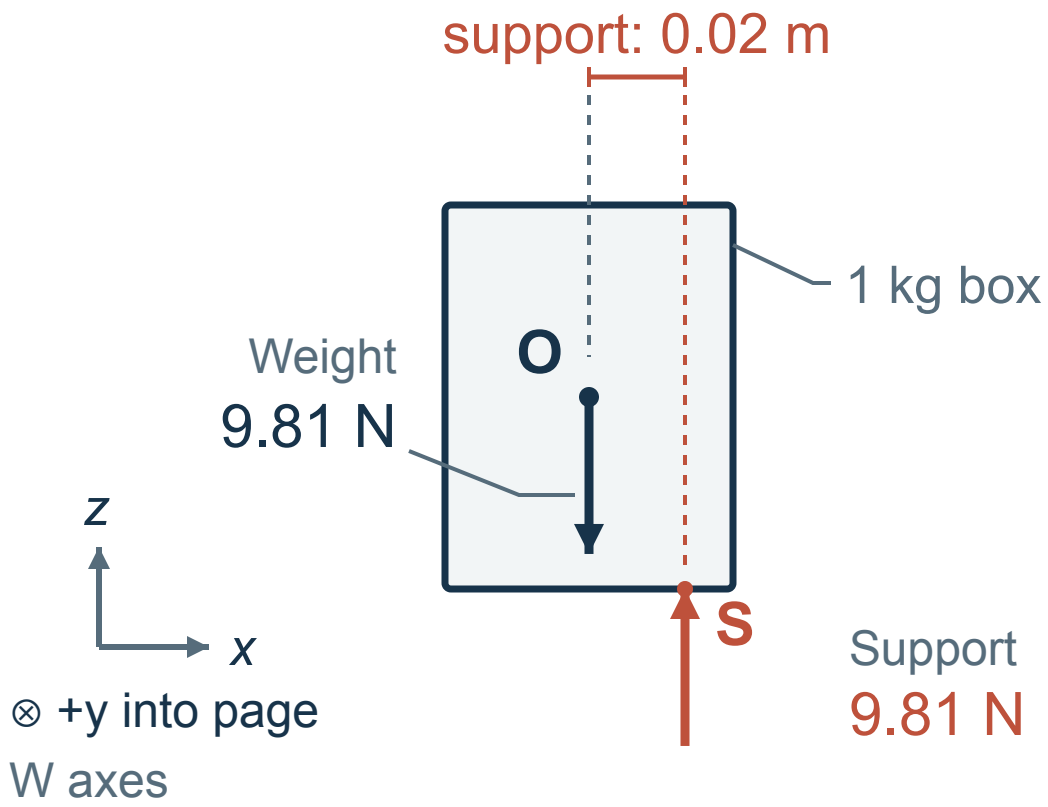


A **force** has a magnitude, direction, and application point.

Its magnitude is measured in newtons (N).

A **free-body diagram** shows the forces acting on this box.

Moment (torque) measures a turning effect



$$\boldsymbol{\tau}_O = {}^{O}\mathbf{p}_W^S \times \mathbf{f}$$

${}^{O}\mathbf{p}_W^S$: vector from O to the force application point S.

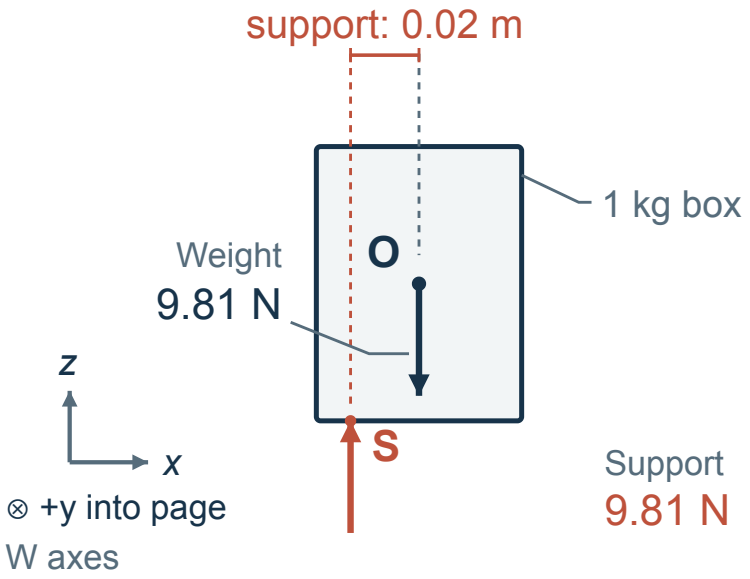
For the upward support force:

$$|\tau_y| = d\, f_{\text{sup}}$$

d: perpendicular distance from O to the support force's line of action.

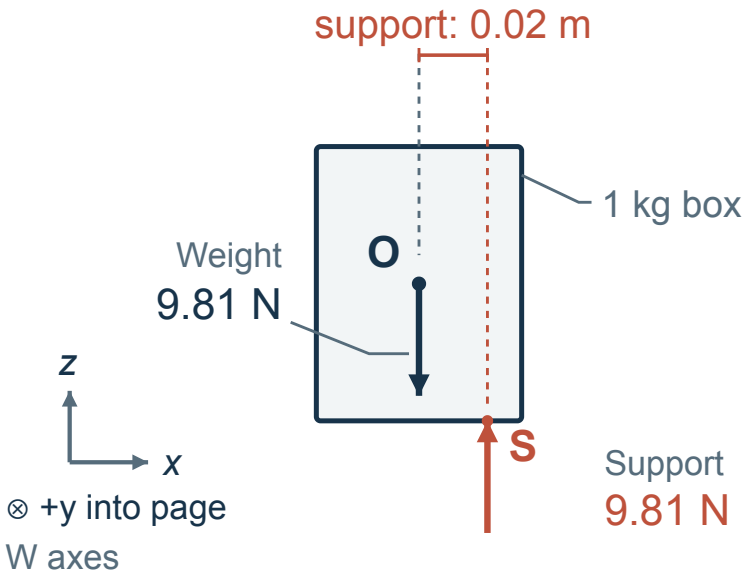
World axes W · position in m · force in N · moment in N m.

Moving the force across O reverses its moment



$$\tau_y = +0.1962\ \mathrm{N\,m}$$

Clockwise

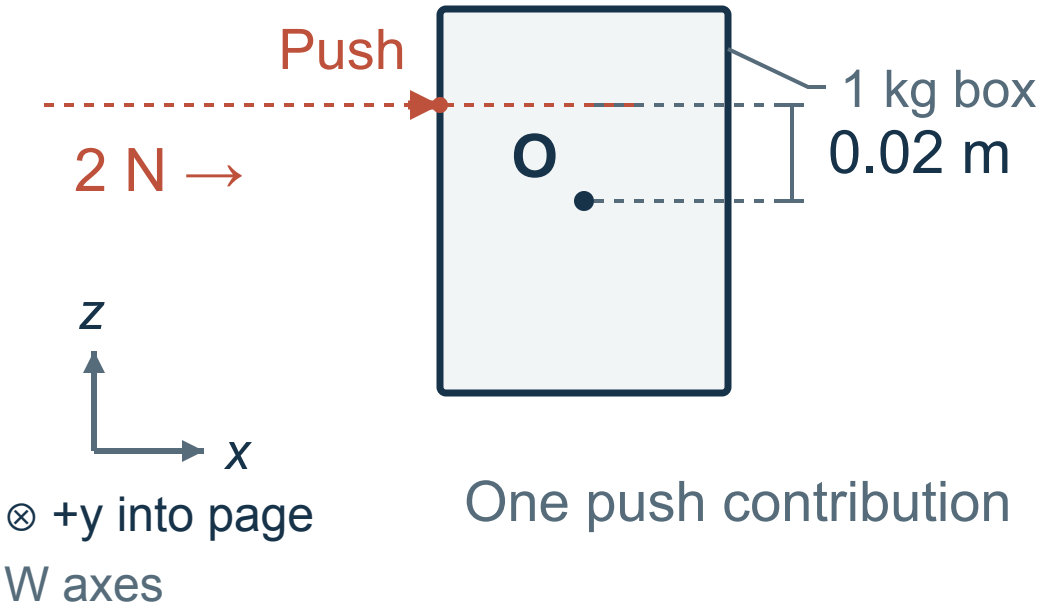


$$\tau_y = -0.1962\ \mathrm{N\,m}$$

Counterclockwise

World y points into the page. Same force magnitude and perpendicular distance.

A horizontal force uses a vertical lever arm



One push contribution

Position from O × force:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} yf_z - zf_y \\ zf_x - xf_z \\ xf_y - yf_x \end{bmatrix}$$

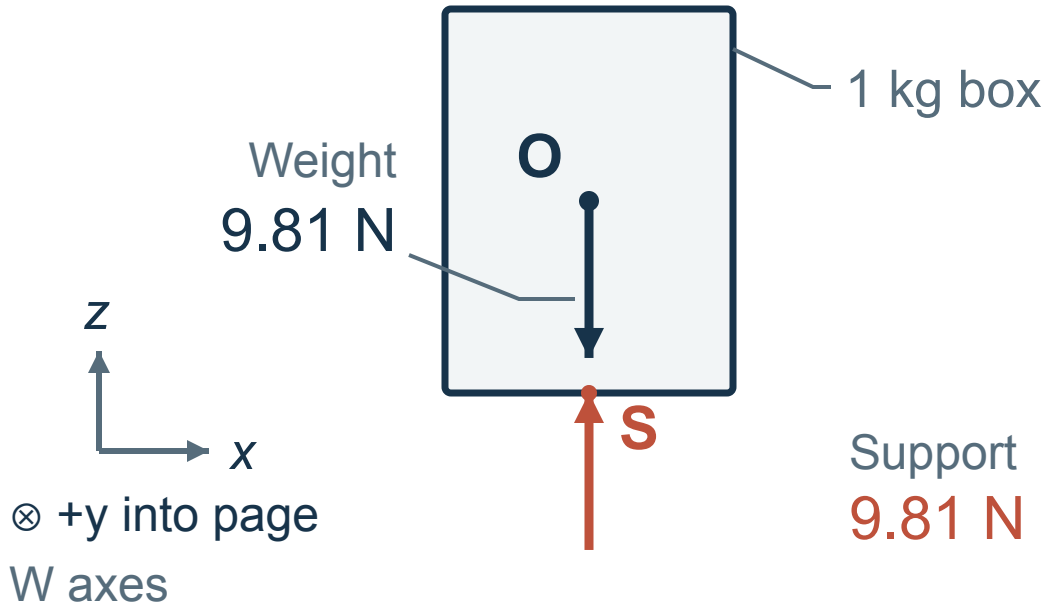
In this plane: $y = 0$, $f_y = 0$.

$$\tau_y = zf_x - xf_z$$

x, y, z: position from O.
f_x, f_y, f_z: force components.

$$\tau_y = +0.04\ \mathrm{N\,m}$$

Which balance changes when we move or strengthen the support?

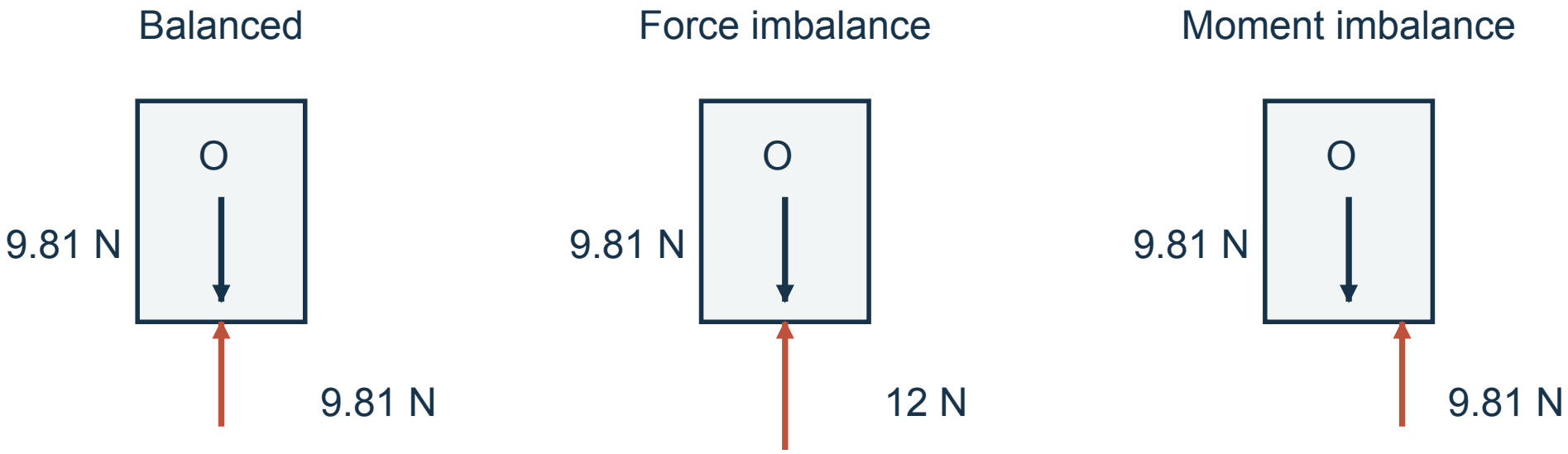


Predict both balances:

- Total force
- Total moment about O

Open live demo ↗

Static equilibrium requires both balances



Both sums zero

Net force upward

Net moment nonzero

$$\sum_i \mathbf{f}_i = \mathbf{0} \qquad \sum_i \boldsymbol{\tau}_O^{(i)} = \mathbf{0}$$

Sum the forces on the same body; report every moment about the same point, in the same axes.

A wrench collects moment and force

On box B · about O · expressed in W

$$\mathbf{F}_{\text{net},W}^{B_O} = \begin{bmatrix} \boldsymbol{\tau}_O \\ \mathbf{f}_{\text{net}} \end{bmatrix}$$

Moment first, then force. Their units are N m and N.

In this plane, keep $[\tau_y, f_x, f_z]^T$.

Changing the reporting point changes the lever arms

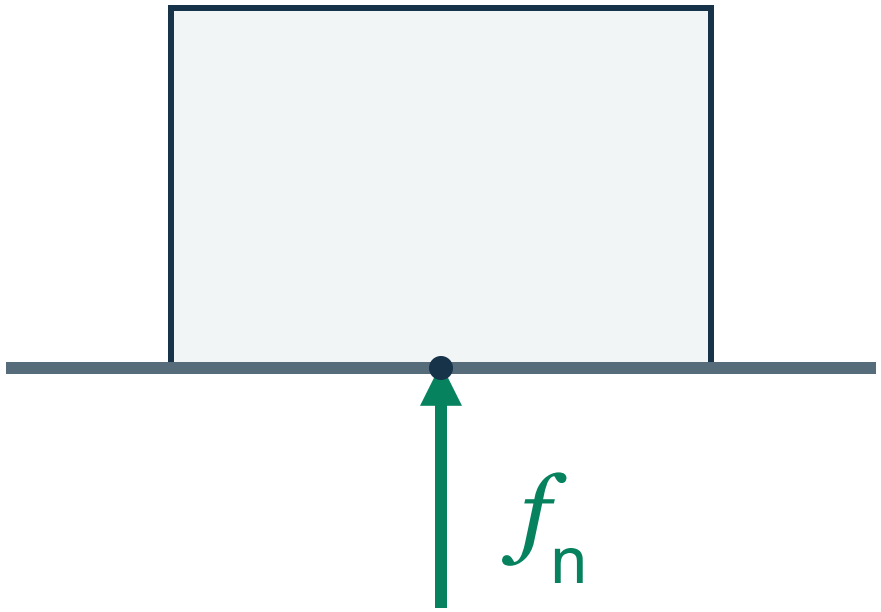
$$\boldsymbol{\tau}_{\text{net},Q} = \boldsymbol{\tau}_{\text{net},O} + {}^{Q}\mathbf{p}_{W}^{O} \times \mathbf{f}_{\text{net}}$$

Q is 0.02 m right of O: ${}^{Q}\mathbf{p}_{W}^{O} = [-0.02, 0, 0]^{\mathsf{T}}$ m. The forces and world axes stay fixed.

Same physical loading	Net upward force	Moment about O	Moment about Q
9.81 N support, 2 cm right	0 N	−0.1962 N m	−0.1962 N m
12 N support, centered	2.19 N	0 N m	+0.0438 N m

Zero net force makes the **total** moment independent of the reporting point.

An ordinary contact can push, but cannot pull



The normal is perpendicular to the surface, positive **into the box**.

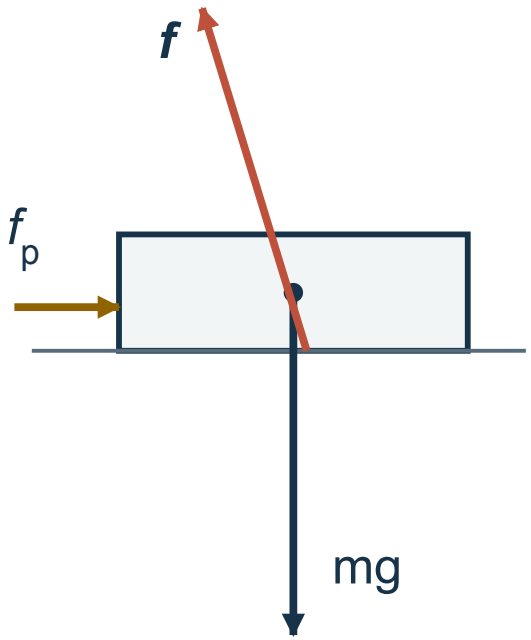
$$f_n \geq 0$$

Separated surfaces supply no contact force.

Contact without adhesion; forces on the box.

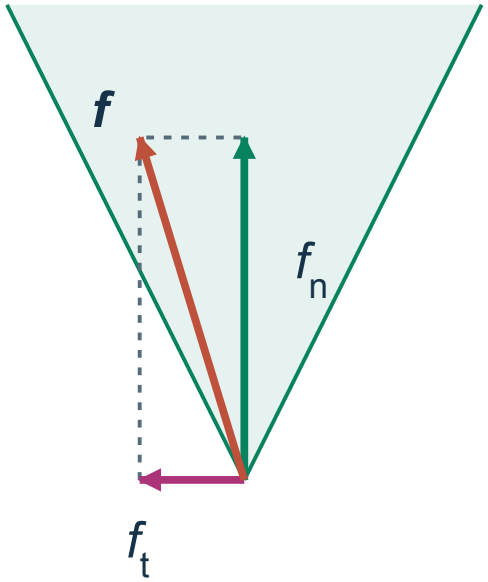
The friction cone contains admissible contact forces

Forces on the box



1 kg · push 3 N

Contact force



same force, resolved

$$\mathbf{f} = f_n \mathbf{n} + f_t \mathbf{t}$$

Unit directions: $\mathbf{n}$ upward, $\mathbf{t}$ rightward.

For rest: $f_n = mg = 9.81\ \mathrm{N}$, $f_t = -f_p = -3\ \mathrm{N}$.

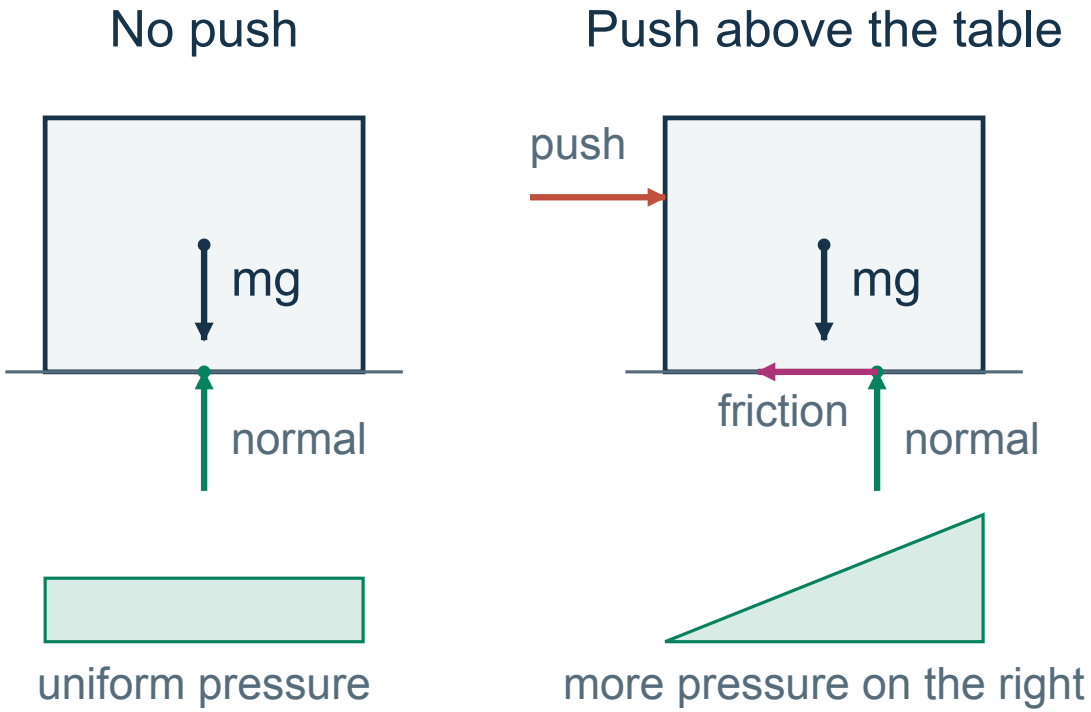
$$|f_t| \le \mu_\mathrm{f} f_n$$

3 N needed; 4.905 N available.

$\mu_\mathrm{f} = 0.5$. The planar cone is a wedge; its cropped height is not a force cap.

Open live demo ↗

The support force can act away from the base center



Pressure is distributed over the contact area.

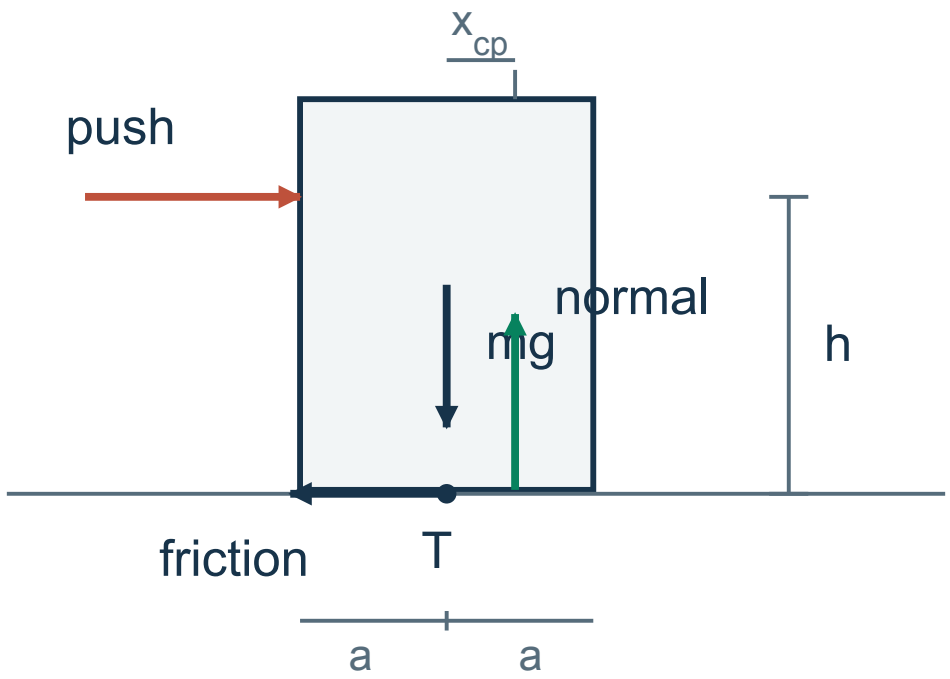
The total upward force acts at the **center of pressure**, x_{cp}.

A push above the table tends to tip the box clockwise.

To balance that moment, the upward support acts farther to the right.

Same total upward force; different application point. Pressure profiles are schematic.

Moment balance determines the required support point



About support midpoint T. At rest: $f_n = mg$, $f_t = -f_p$.

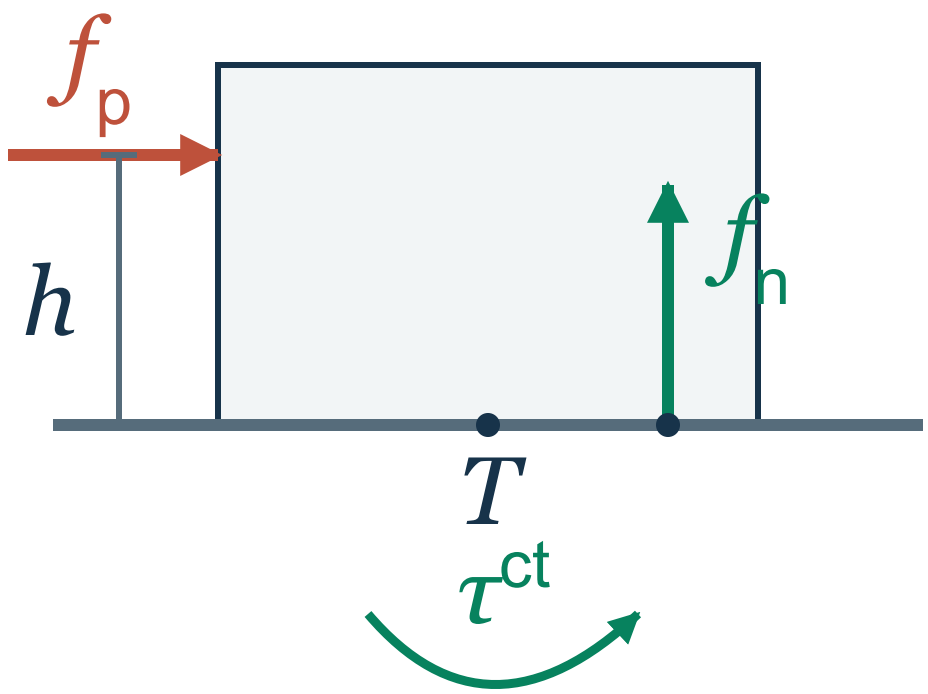
$$hf_p - x_{\text{cp}} mg = 0$$

$$x_{\text{cp}} = \frac{hf_p}{mg}$$

A stronger or higher push needs support farther toward the edge.

h: push height; a: support half-width; x_{cp}: normal-resultant location from T.

A finite base limits the balancing moment



$$\tau_{y,T}^{\mathrm{ct}} = -x_{\mathrm{cp}} f_n$$

$$|x_{\mathrm{cp}}| \le a$$

$$\Longrightarrow \quad |\tau_{y,T}^{\mathrm{ct}}| \le a f_n$$

The required support point must lie inside the base.

Two descriptions of the same flat support limit.

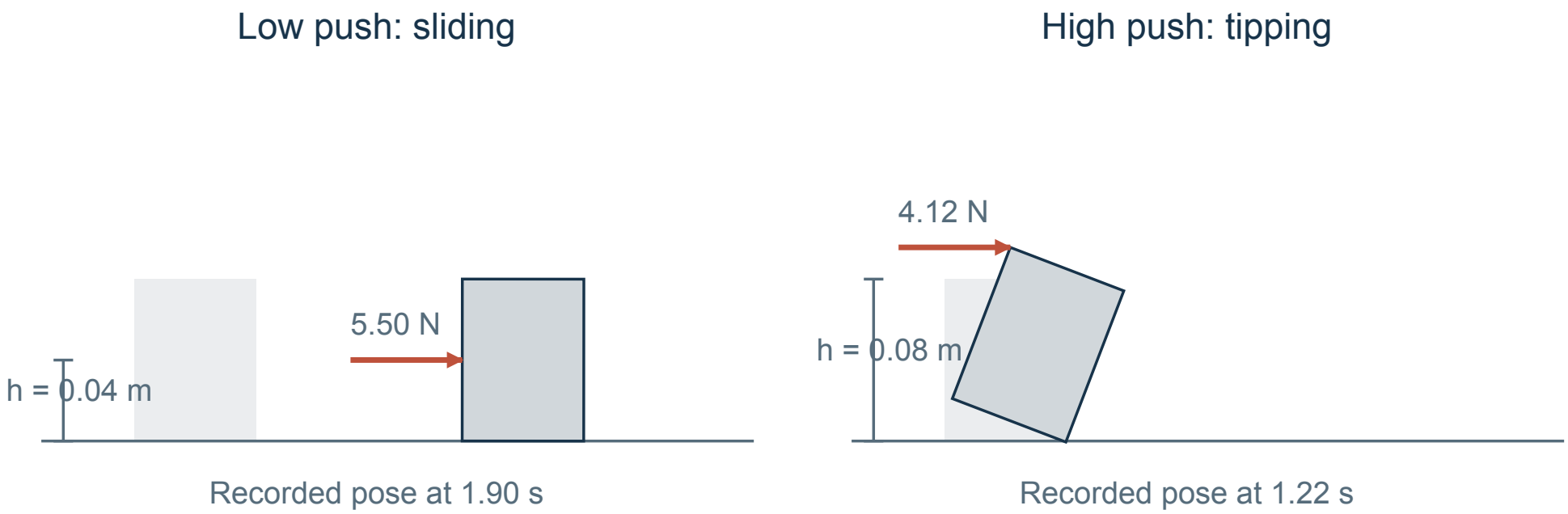
Which limit is reached first?

$$f_{\text{slide}} = \mu_{\text{f}} m g \qquad f_{\text{tip}} = \frac{a m g}{h}$$

Push height	Sliding threshold	Tipping threshold
0.04 m	4.905 N	7.358 N
0.06 m	4.905 N	4.905 N
0.08 m	4.905 N	3.679 N

Same 1 kg box, support half-width 0.03 m, friction coefficient 0.5.

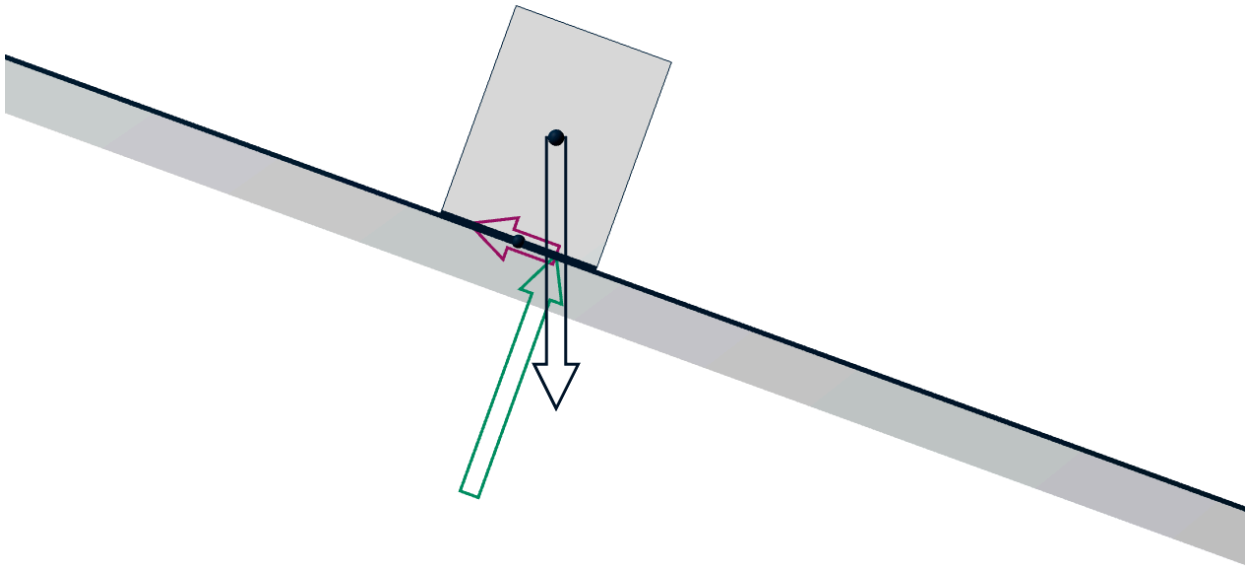
Push height changes sliding into tipping



h: initial push height · pale box: initial pose · applied push shown

Same 1 kg box and friction coefficient. Compare which contact limit is exhausted.

The normal follows the surface



θ: ramp angle below horizontal

At rest · positive f_t downhill

$$f_n = mg\cos\theta$$

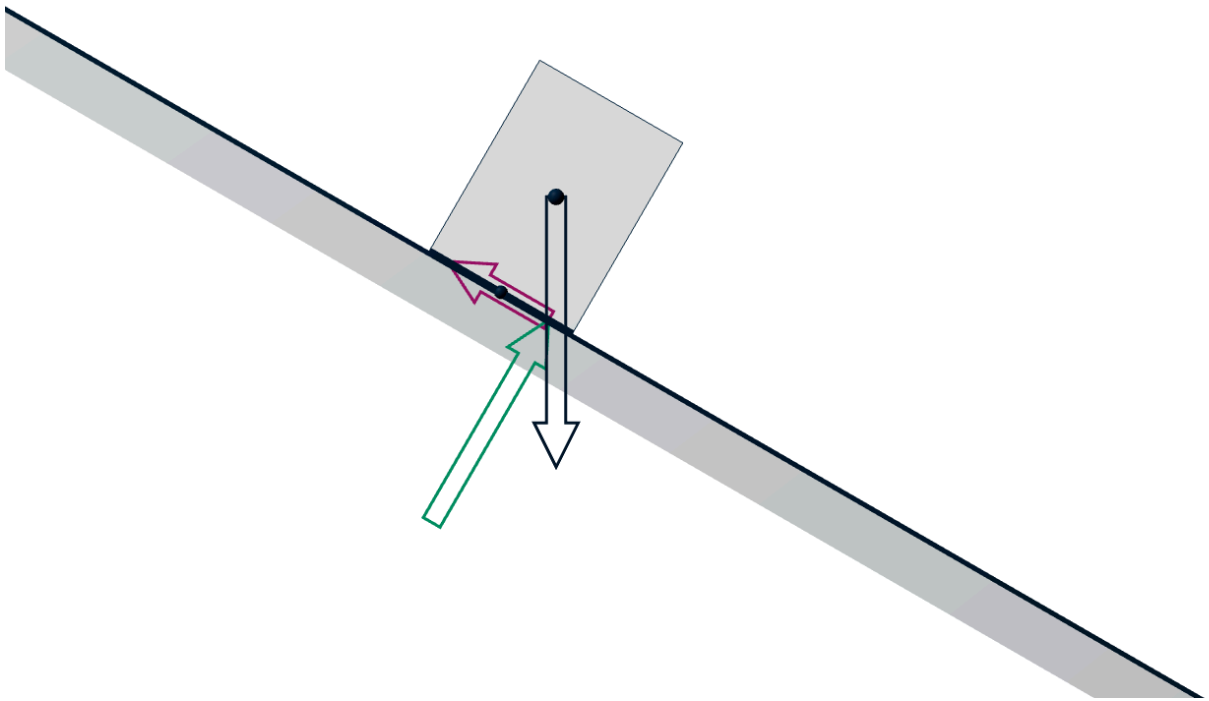
$$f_t = -mg\sin\theta$$

Green: normal · purple: friction

Navy: weight on the box

Drake · 1 kg · 20° · $\mu_\mathrm{f} = 0.5$

A steeper ramp needs more friction



Friction condition for rest

Stationary ramp

$$mg\sin\theta \le \mu_{\mathrm{f}} mg\cos\theta$$

$$\Rightarrow \quad \tan\theta \le \mu_{\mathrm{f}}$$

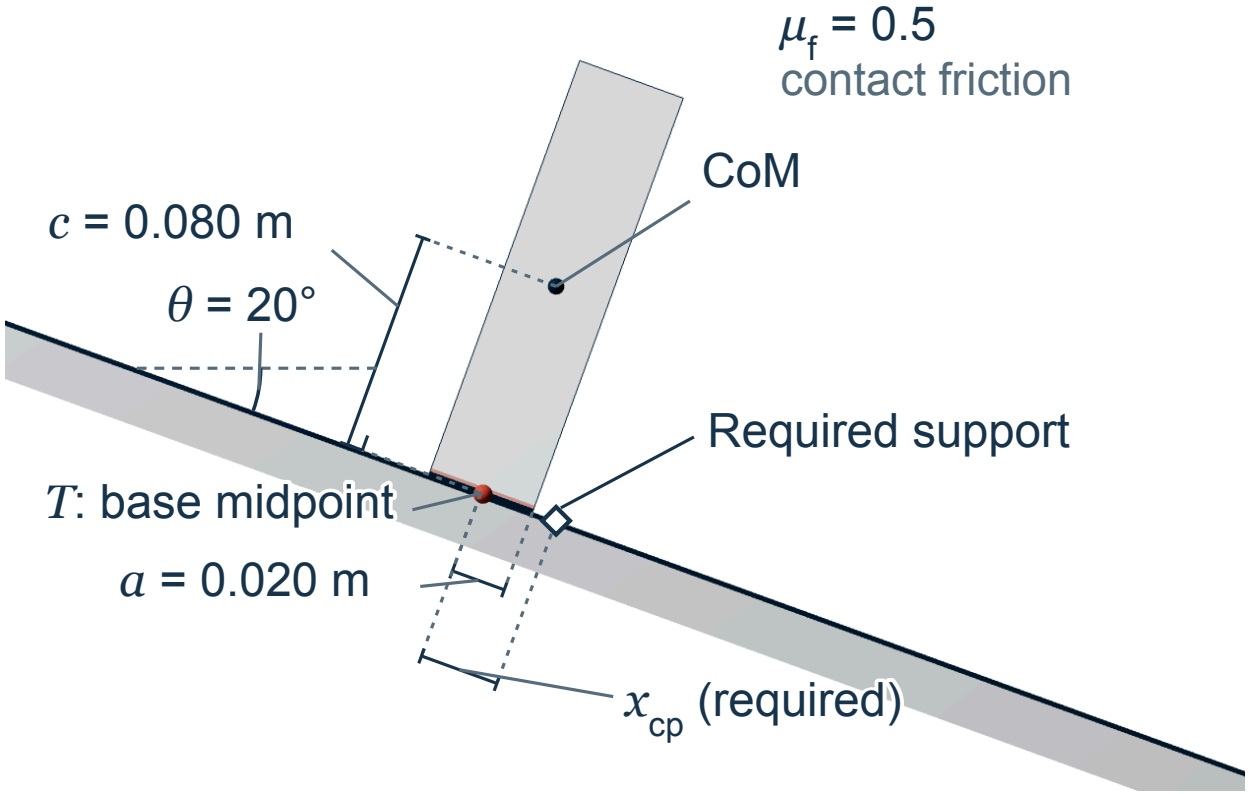
At $\mu_{\mathrm{f}} = 0.5$: limit 26.6°.

More mass scales both forces;

the ideal angle stays the same.

Drake · 1 kg · 30° · $\mu_{\mathrm{f}} = 0.5$

Enough friction can still leave a tipping moment



Flat table: moments about T

$$h f_p = x_{\text{cp}} mg$$

Ramp: $mg\sin\theta$ acts downhill;

normal support must supply $mg\cos\theta$.

$$c\,mg\sin\theta = x_{\text{cp}}\,mg\cos\theta$$

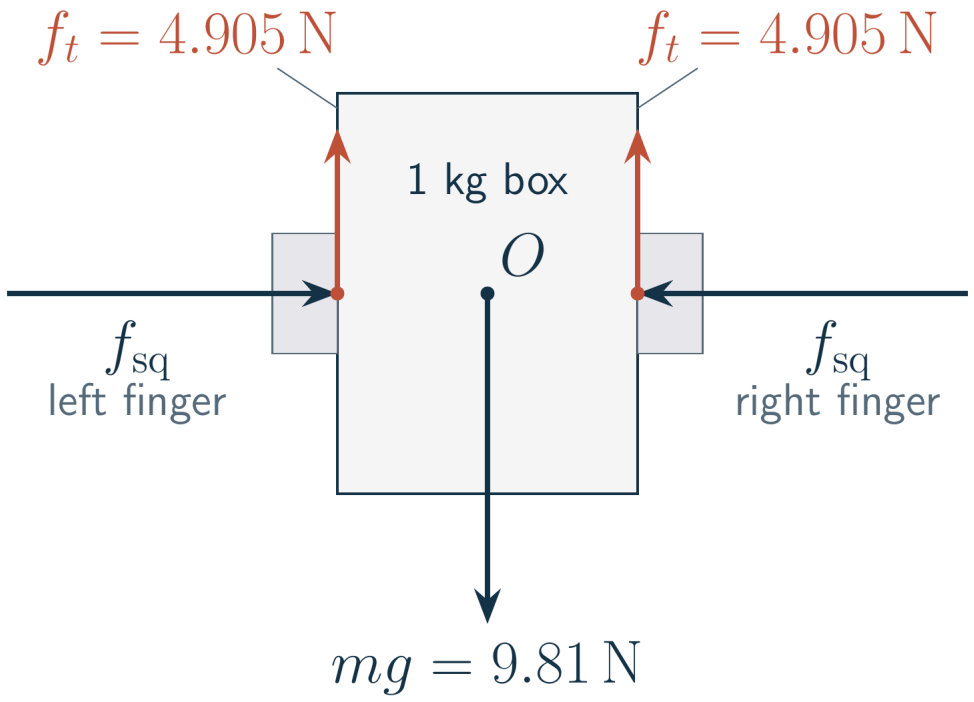
$$x_{\text{cp}} = c\frac{\sin\theta}{\cos\theta} = c\tan\theta$$

For equilibrium: $c\tan\theta \le a$.

At 20°: **0.0291 m > 0.020 m**.

Required support lies beyond the base.

In this symmetric grasp, each finger supports half the weight



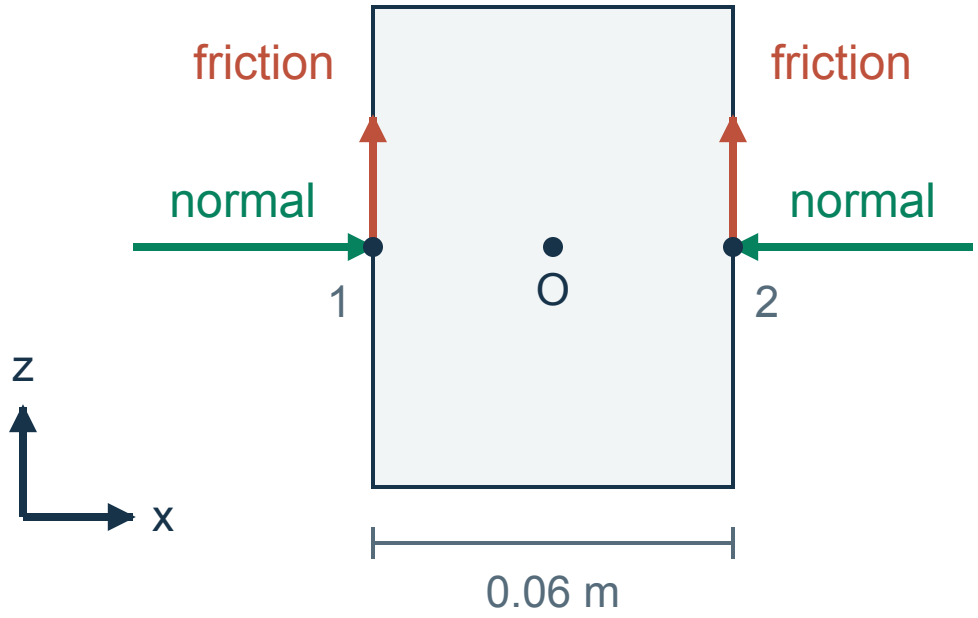
Forces on the box · $\mu_{\mathrm{f}} = 0.5$

$$2f_t = mg \quad \Rightarrow \quad f_t = 4.905\ \mathrm{N}$$

$$f_{\mathrm{sq}} \geq \frac{mg}{2\mu_{\mathrm{f}}} = 9.81\ \mathrm{N}$$

Minimum compression **per finger**.

The grasp map combines the contact forces



$$\mathbf{F}_{\mathrm{ct}} = \mathbf{G}_{\mathrm{grasp}}\,\mathbf{f}_{\mathrm{c}}$$

Stacked contact-force vector

$\mathbf{f}_{\mathrm{c}} = [f_{n,1}, f_{t,1}, f_{n,2}, f_{t,2}]^{\mathsf{T}}$

Resultant contact wrench

$\mathbf{F}_{\mathrm{ct}} = [\tau_y, f_x, f_z]^{\mathsf{T}}$

Grasp matrix (grasp map)

$\mathbf{G}_{\mathrm{grasp}}: 3 \times 4.$

Each **column of it** describes the wrench produced by one unit of the corresponding contact-force component.

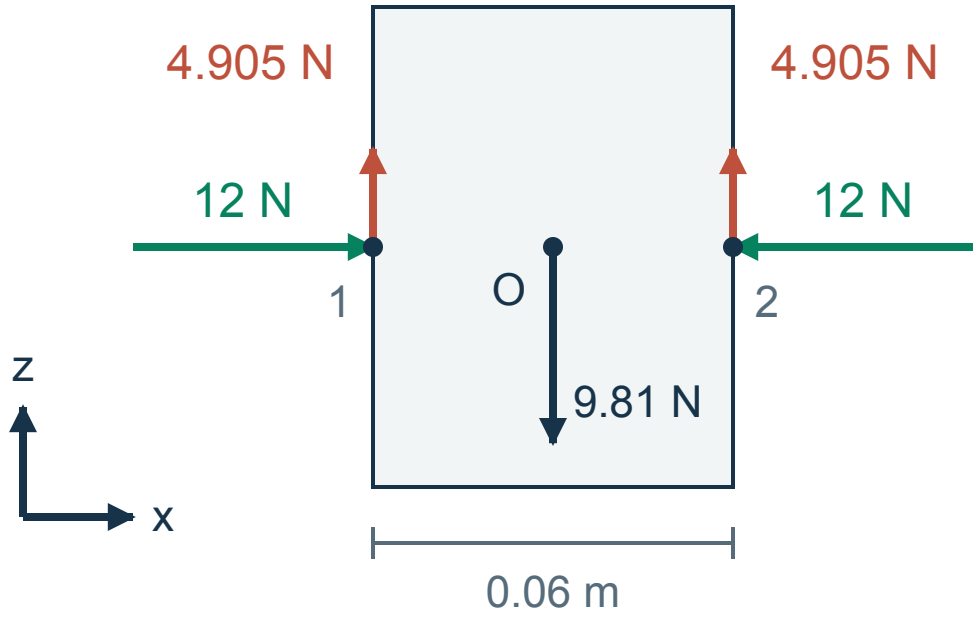
Build the grasp matrix one column at a time

Contacts: $x = \pm a$, $z = 0$, $a = 0.03\,\mathrm{m}$. **Moment from force k about O:** $\tau_y^{(k)} = z_k f_{x,k} - x_k f_{z,k}$.

Wrench entry	Left normal $f_{n,1}$	Left tangent $f_{t,1}$	Right normal $f_{n,2}$	Right tangent $f_{t,2}$
$\tau_y^{(k)}$	0	$+a f_{t,1}$	0	$-a f_{t,2}$
f_x	$+f_{n,1}$	0	$-f_{n,2}$	0
f_z	0	$+f_{t,1}$	0	$+f_{t,2}$

$$\underbrace{\begin{bmatrix}\tau_y \\ f_x \\ f_z\end{bmatrix}}_{\mathbf{F}_{\mathrm{ct}}} = \underbrace{\begin{bmatrix}0 & a & 0 & -a \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1\end{bmatrix}}_{\mathbf{G}_{\mathrm{grasp}}} \underbrace{\begin{bmatrix}f_{n,1} \\ f_{t,1} \\ f_{n,2} \\ f_{t,2}\end{bmatrix}}_{\mathbf{f}_{\mathrm{c}}}$$

Holding a load requires balance and enough friction



$$\mathbf{G}_{\text{grasp}}\mathbf{f}_{\text{c}} + \mathbf{F}_g = \mathbf{0}$$

The forces and moments balance.

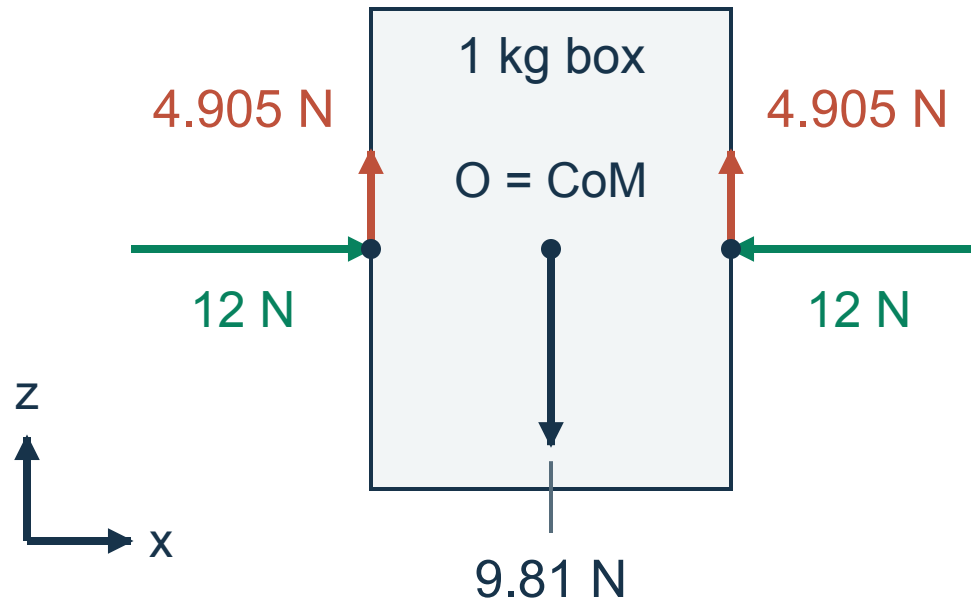
$$4.905\ \text{N} < 0.5(12\ \text{N}) = 6\ \text{N}$$

Each contact has enough friction.

$\mathbf{F}_g$: gravity's wrench about O.

Centered 1 kg box · $\mu_{\text{f}} = 0.5$ · 12 N compression per finger.

More squeeze raises the friction limit



Equal extra squeeze cancels on the box.

$$f_{\mathrm{sq}} \uparrow \quad \Longrightarrow \quad \mu_{\mathrm{f}} f_{\mathrm{sq}} \uparrow$$

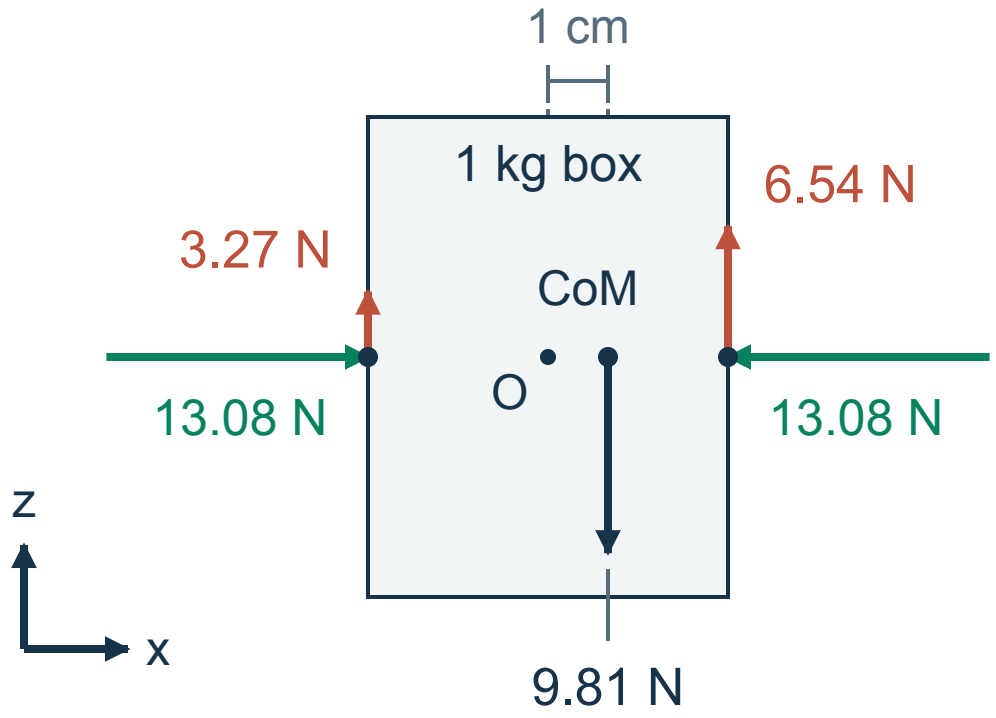
At 12 N squeeze per finger:

6 N available; 4.905 N needed upward.

f_{sq}: inward squeeze force per finger.

Centered 1 kg box · $\mu_{\mathrm{f}} = 0.5$ · forces on the box, moments about O.

An off-center load needs unequal upward forces



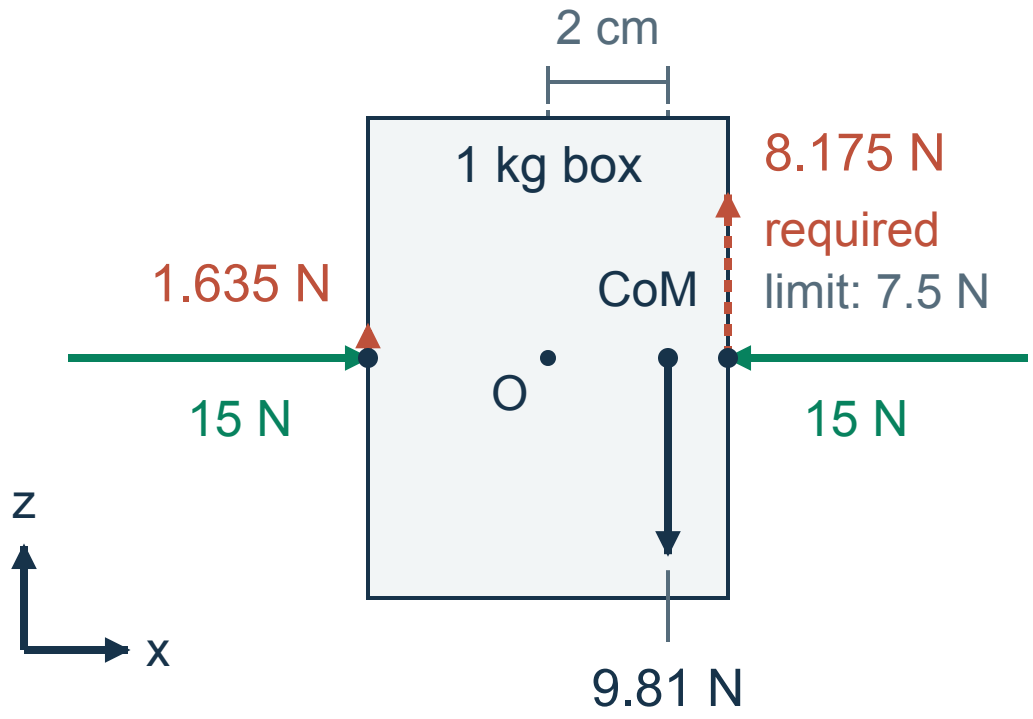
$$f_{t,1} + f_{t,2} = mg$$

$$a(f_{t,1} - f_{t,2}) + mgx_{\text{com}} = 0$$

$$f_{t,1} = 3.27\ \text{N}, \qquad f_{t,2} = 6.54\ \text{N}$$

$x_{\text{com}} = 0.01$ m; $a = 0.03$ m. Moments about O, midway between the contacts. About the CoM, the weight term drops out; the forces are the same.

A force cap can make this load infeasible



Required squeeze exceeds the cap:

$$16.35\ \mathrm{N} > 15\ \mathrm{N}$$

Right contact: **8.175 N needed**, only **7.5 N available** upward.

These two fixed point contacts cannot balance this load.

CoM 2 cm right of O · $\mu_\mathrm{f} = 0.5$ · dashed arrows: forces required for balance.

All three constraints must hold simultaneously

Fixed geometry, planar friction → **linear program (LP)**

$$\min_{\mathbf{f}_c,\, s} \; s$$

s: compression ceiling for both fingers (N).

Find the smallest ceiling that still allows balance and sufficient friction.

Balance $$\mathbf{G}_{\text{grasp}}\mathbf{f}_c + \mathbf{F}_g = \mathbf{0}$$

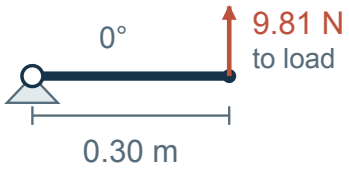
Friction $$|f_{t,i}| \le \mu_f f_{n,i}$$

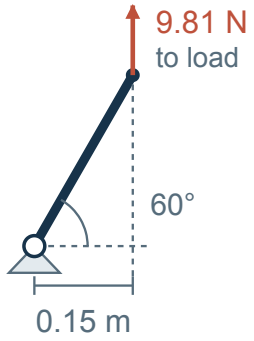
Compression $$0 \le f_{n,i} \le s, \qquad f_{n,i} \le 15\ \text{N}$$

> **3D, exact circular friction cones:**
> convex second-order cone program (SOCP), not LP.

The same force can require different joint torques

Supporting a **1 kg object**: $mg = 9.81\ \mathrm{N}$.





$$|\tau_{\text{joint,load}}| = 2.943\ \mathrm{N\,m} \qquad |\tau_{\text{joint,load}}| = 1.4715\ \mathrm{N\,m}$$

$$\boldsymbol{\tau}_{\text{joint,load}} = \mathbf{J}_c^{\mathsf{T}}\mathbf{f}_{\text{tip}}$$

Why the transpose? Matching power.

$$\mathbf{f}_{\text{tip}}^{\mathsf{T}}\mathbf{v}_{\text{tip}} = \boldsymbol{\tau}_{\text{joint,load}}^{\mathsf{T}}\dot{\mathbf{q}}$$

Endpoint power = joint power (W).

$$\mathbf{v}_{\text{tip}} = \mathbf{J}_c\dot{\mathbf{q}}$$

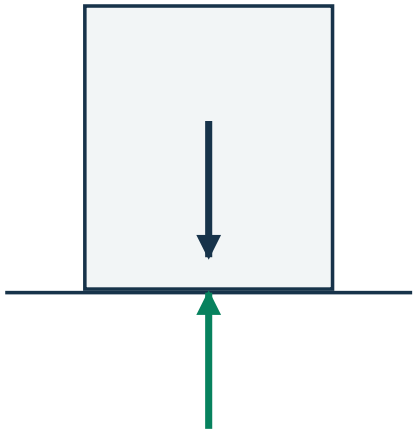
$\mathbf{J}_c$ maps joint velocities to endpoint linear velocity.

$\mathbf{f}_{\text{tip}}$: force exerted on the load; $\mathbf{v}_{\text{tip}}$: velocity at the same point, in the same coordinates.

Joint effort to balance contact loading; robot self-weight excluded.

Supporting gravity does not imply force closure

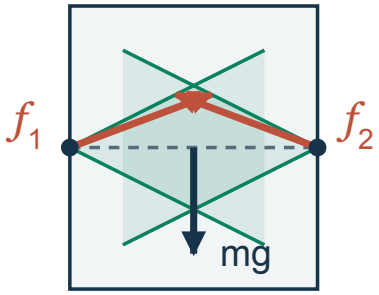
Frictionless table



Holds gravity

Cannot oppose a sideways load

Opposed frictional points



Shown: force AND moment balance

Equal upward components support the weight

Planar force closure for $\mu_f > 0$

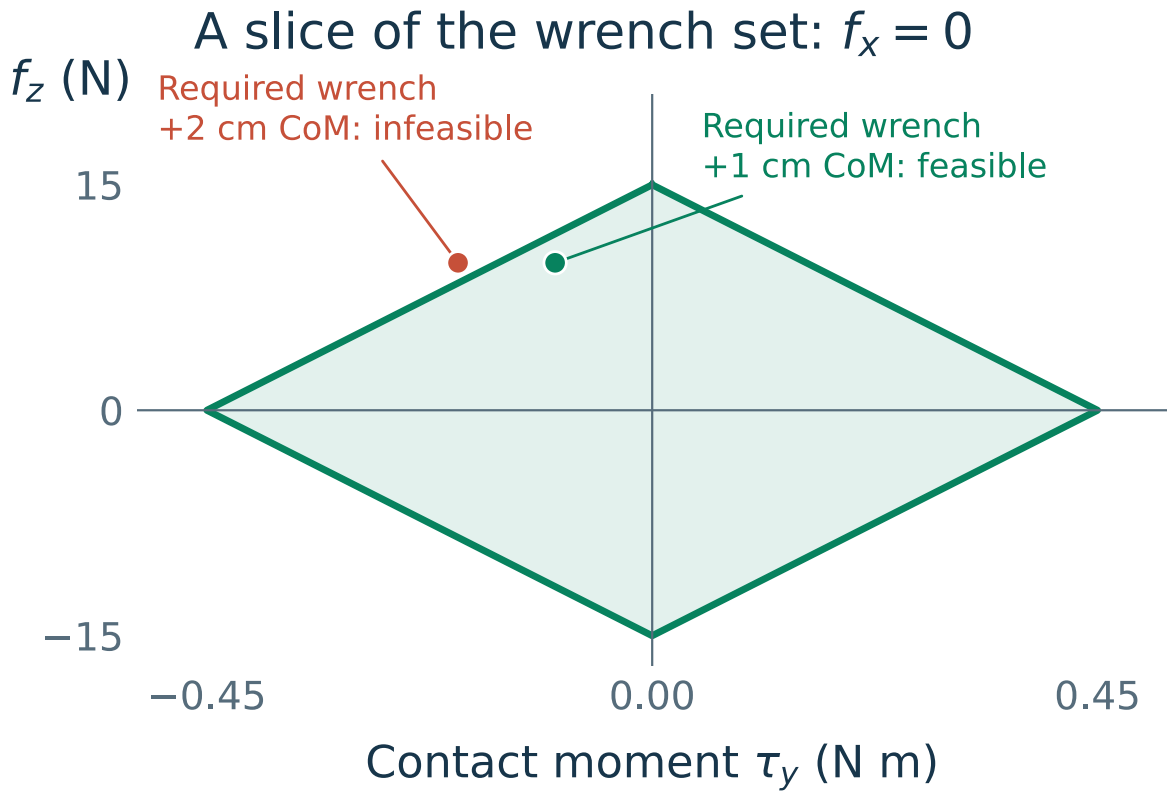
Can also oppose forces and moments in every planar direction

Force closure: contact forces can be chosen to oppose **every force and moment direction**.

One balanced allocation is shown; closure concerns all available allocations.

Planar point contacts · sufficient squeeze · no force caps.

Force limits restrict which loads a grasp can support



Achievable contact-wrench set $\mathcal{W}_{\text{grasp}}$

... within friction and finger-force limits.

$$\mathcal{W}_{\text{grasp}} = \{\mathbf{G}_{\text{grasp}}\mathbf{f}_{\text{c}} : \text{contact limits hold}\}$$

$$-\mathbf{F}_{\text{ext}} \in \mathcal{W}_{\text{grasp}}$$

To balance the external load, the contacts must produce its **opposite wrench** — inside the set or on its boundary.

Same pinch · $\mu_{\text{f}} = 0.5$ · 15 N compression cap per finger.

Scale force and moment before measuring distance

> To score grasp capacity using distance, we first need comparable force and moment coordinates.

$$\widehat{\mathbf{F}} = \begin{bmatrix} \tau_y / (F_{\mathrm{ref}} \ell_{\mathrm{ref}}) \\ f_x / F_{\mathrm{ref}} \\ f_z / F_{\mathrm{ref}} \end{bmatrix}$$

Chosen reference scales

$F_{\mathrm{ref}} = 9.81\ \mathrm{N}$: the 1 kg object's weight.

$\ell_{\mathrm{ref}} = 0.03\ \mathrm{m}$: half the contact spacing.

The normalized components are dimensionless.

S25's 1 kg box: CoM 1 cm right of O, midway between the contacts.

Moments about O: gravity $+0.0981$ N m; required contact moment -0.0981 N m.

$$\widehat{\mathbf{F}}_{\mathrm{required}} = \begin{bmatrix} -0.0981/(9.81 \times 0.03) \\ 0/9.81 \\ 9.81/9.81 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

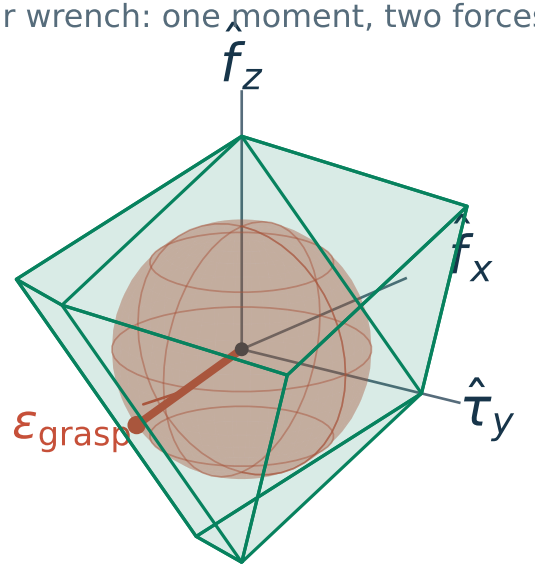
Use the same scales for comparisons. Scaling preserves load feasibility.

Score the grasp by its weakest wrench direction

> Two grasps can both hold the weight. Which has greater capacity when load direction is uncertain?

Normalized achievable contact-wrench set

Planar wrench: one moment, two forces



Ball centered at zero wrench

Compare candidate grasps using their capacity in the weakest normalized wrench direction.

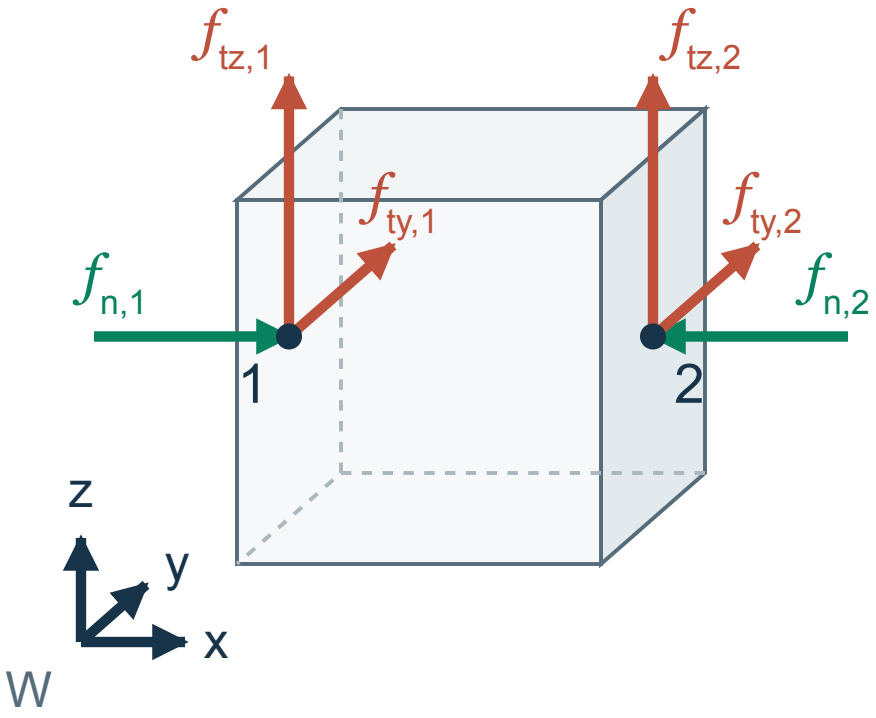
ϵ_{grasp}: radius of the largest ball **centered at zero wrench** inside the achievable set.

Larger radius → greater weakest-direction capacity.

Use the same scales and force limits for comparisons.

For a specific gravity-plus-task load, test its opposing wrench against the **full set**. This radius is not the remaining margin while holding gravity.

In 3D, tangential forces share one friction limit



One contact, one tangential friction force — resolved into two components.

Compression limits the force's **total magnitude**.

$$\sqrt{f_{t_y,i}^2 + f_{t_z,i}^2} \leq \mu_\mathrm{f} f_{n,i}$$

$\mu_\mathrm{f} = 0.5$, $f_{n,i} = 15$ N → **7.5 N limit**.

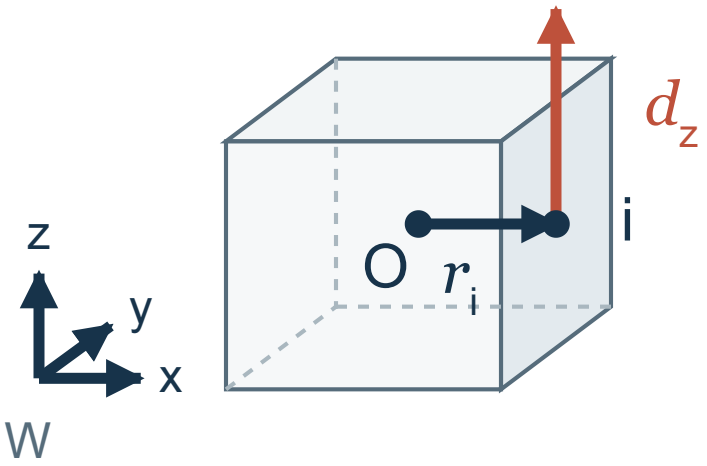
Two 6 N tangential components require

$$\sqrt{6^2 + 6^2} = 8.49 \text{ N} > 7.5 \text{ N}.$$

Isotropic Coulomb friction: same limit in all surface directions. Tangents: world y, z; contact index: i.

Extend grasp-matrix construction to 3D

> As on S22, factor out each scalar force component. The remaining coefficients form one matrix column.



$\mathbf{r}_i$: from O to contact i.
$\mathbf{d}_z = [0, 0, 1]^\mathsf{T}$: upward unit direction.

For the upward component alone, $\mathbf{f}_i = \mathbf{d}_z f_{t_z,i}$:

$$\mathbf{F}_i = \underbrace{\begin{bmatrix} \mathbf{r}_i \times \mathbf{d}_z \\ \mathbf{d}_z \end{bmatrix}}_{\text{one column of } \mathbf{G}_{\text{grasp}}} \underbrace{f_{t_z,i}}_{\text{force component (N)}}$$

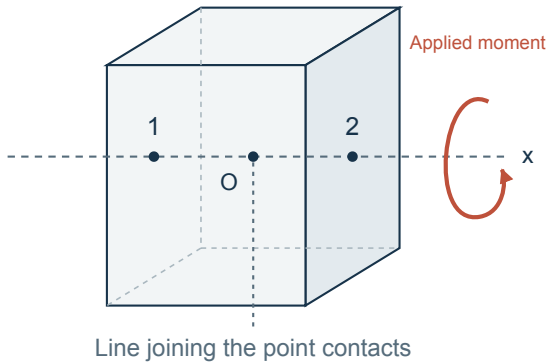
Same construction; now **3 moment components and 3 force components**.

6 columns: 3 directions × 2 contacts. **6 rows:** 3 moments + 3 forces.

Moments about O · all vectors in world coordinates · fixed contact geometry.

Two point contacts leave one moment direction unavailable

> Six force components—but can they produce every spatial wrench direction?



Ideal point contacts: $x = -a, +a$; $y = z = 0$.
Moments about O; world coordinates.

$$\tau_{x,i} = y_i f_{z,i} - z_i f_{y,i} = 0$$

No lever arm for torque about this line. **More squeeze cannot help.**

Full spatial grasp map · moments first

$$\underbrace{\begin{bmatrix} \tau_x^{\mathrm{ct}} \\ \tau_y^{\mathrm{ct}} \\ \tau_z^{\mathrm{ct}} \\ f_x^{\mathrm{ct}} \\ f_y^{\mathrm{ct}} \\ f_z^{\mathrm{ct}} \end{bmatrix}}_{\mathbf{F}_{\mathrm{ct}}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & -a \\ 0 & -a & 0 & 0 & a & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{G}_{\mathrm{grasp}}} \underbrace{\begin{bmatrix} f_{n,1} \\ f_{t_y,1} \\ f_{t_z,1} \\ f_{n,2} \\ f_{t_y,2} \\ f_{t_z,2} \end{bmatrix}}_{\mathbf{f}_{\mathrm{c}}}$$

Normal components point inward.
Positive tangential components point along $+y$, $+z$ at both contacts.

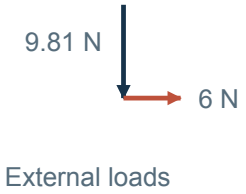
Zero first row: $\tau_x^{\mathrm{ct}} = 0$ for every force choice.

No opposing moment about the contact line → no spatial force closure.

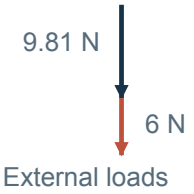
Equal-magnitude added loads can require different friction

> The grasp holds the object. Will an extra 6 N sideways or downward exceed its friction capacity?

1 kg object · 15 N compression per pad · $\mu_f = 0.5 \rightarrow$ 7.5 N tangential limit per pad.
Added forces act at the centered CoM. The symmetric grasp shares the load equally.



External loads



External loads

Required tangential force per pad — oppose half the combined external load

6 N sideways, along $+y$

$$\|\mathbf{f}_t\| = \sqrt{(6/2)^2 + (9.81/2)^2}$$
$$\approx 5.75\ \text{N} < 7.5\ \text{N}$$

Within friction capacity
Perpendicular components combine by magnitude.

6 N downward, along $-z$

$$\|\mathbf{f}_t\| = (9.81 + 6)/2$$
$$= 7.905\ \text{N} > 7.5\ \text{N}$$

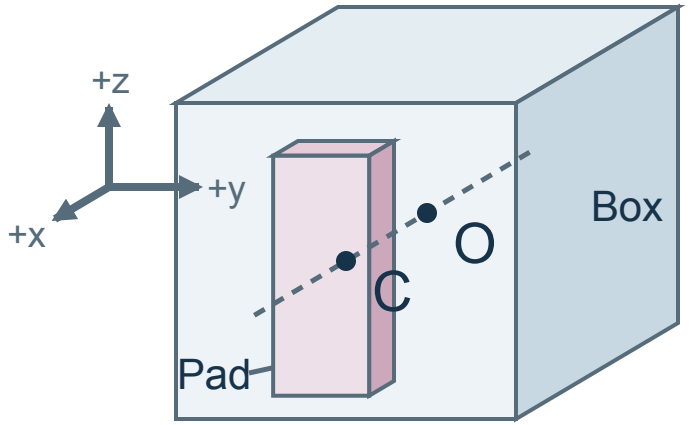
Exceeds friction capacity
Downward forces add directly.

The demo uses finite contact pads. Here we test force capacity; next, their ability to resist twist.

Friction across a pad can produce a resisting moment

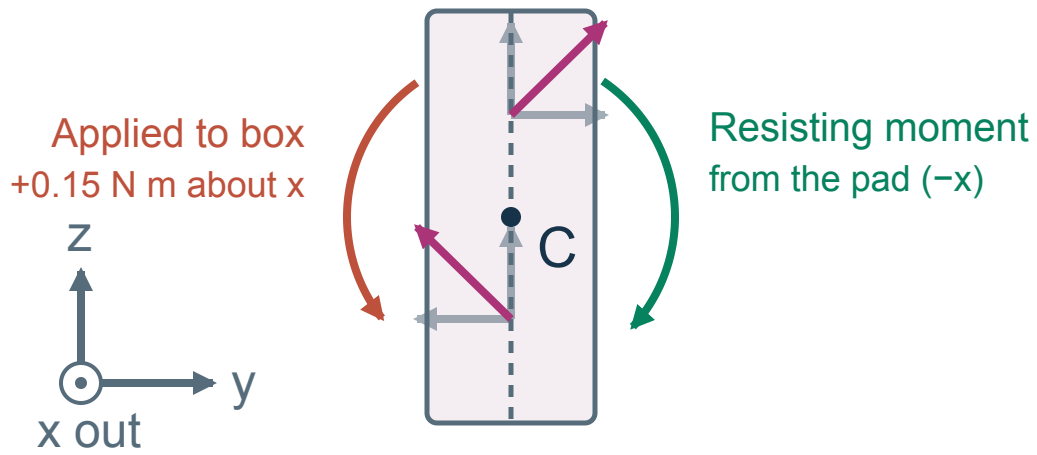
> Point contacts (S35) could not resist axial twist. What changes when contact covers an area?

One pad touching the box



Jaw axis x passes through O and C

Contact face viewed from +x



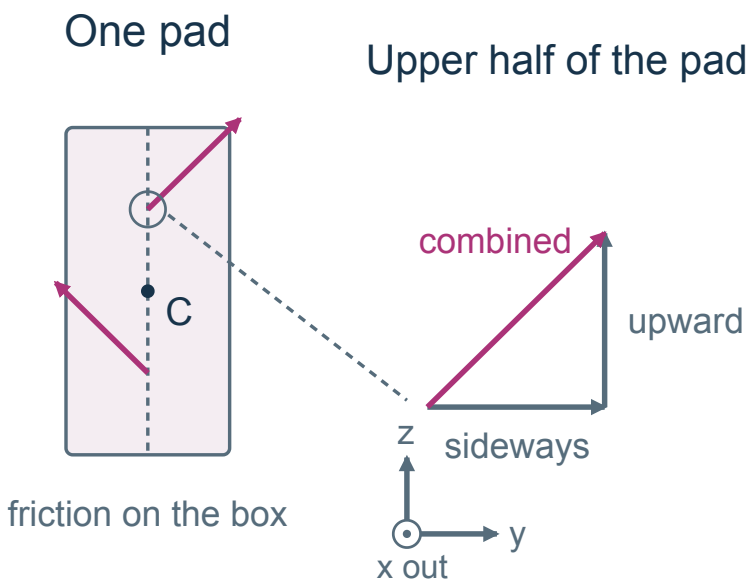
Each purple arrow sums friction over half the pad.

A counterclockwise applied moment tends to move the box surface left above C and right below C relative to the pad.

Friction on the box opposes that slip tendency. **The sideways forces cancel, but their clockwise moments add.** The upward components support the weight.

Supporting weight and resisting twist share friction

> Chosen test load (S37): $+0.15\ \mathrm{N\,m}$ about the jaw axis. Can the pads balance it while supporting the box?



Two pads × two halves. Each half's center is 0.015 m from the jaw axis. The four sideways forces produce moments in the same direction.

$$\sqrt{(\Delta f_y)^2 + (\Delta f_z)^2} \le \mu_\mathrm{f} \Delta f_n$$

Available per half · Uniform pressure splits 15 N per pad equally. Friction limit: $0.5 \times (15/2) = 3.75\ \mathrm{N}$.

Required per half · Four regions share the load equally.

Upward for the 1 kg box: $9.81/4 \approx 2.45\ \mathrm{N}$.

Sideways for a $0.15\ \mathrm{N\,m}$ couple: $0.15/(4 \times 0.015) = 2.50\ \mathrm{N}$.

Combined demand fits the limit:

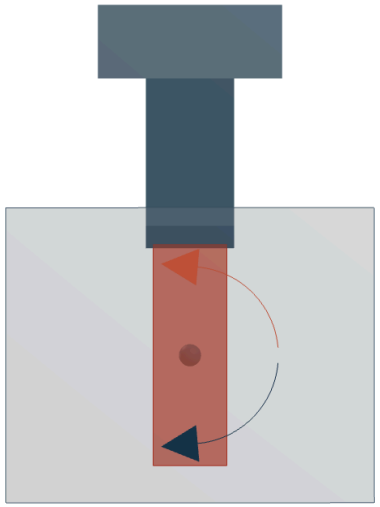
$$\sqrt{2.50^2 + 2.45^2} \approx 3.50\ \mathrm{N} < 3.75\ \mathrm{N}.$$

At fixed squeeze, a larger applied moment can rotate the box

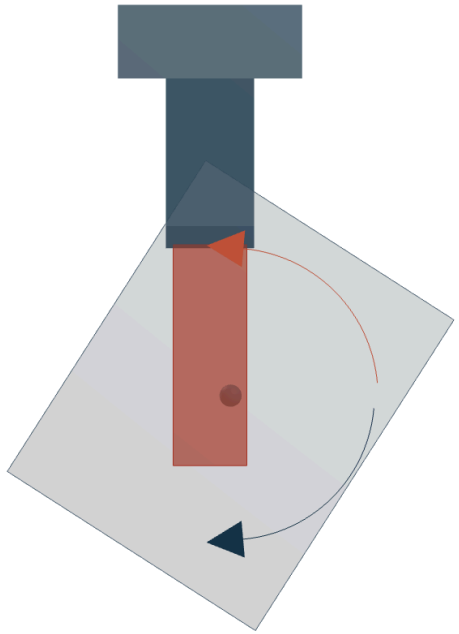
> Keep the squeeze at 15 N per jaw. Will the box stay level if we increase the applied moment from 0.15 to 0.25 N·m?

Same 1 kg box · $\mu_f = 0.5$ · finite contact pads

Applied +0.15 N·m · box stays nearly level



Applied +0.25 N·m · box visibly rotates



Applied moment Contact moment Drake simulation · jaw-axis view · +x toward us

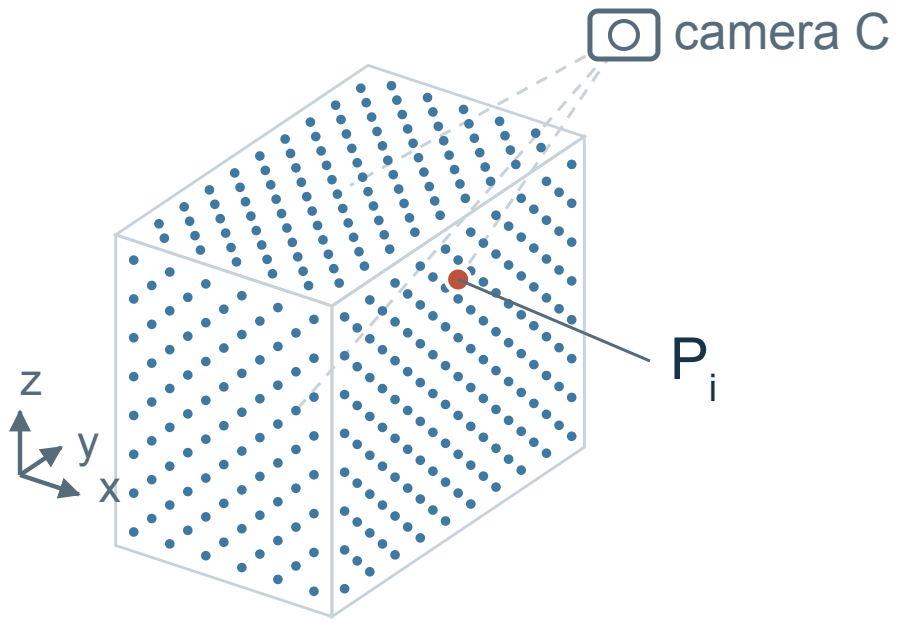
Can a candidate grasp support the task load?

1. **Specify the task load and contact model.** Locate the contacts and choose a point or patch model.
2. **Build the grasp map.** Map the modeled contact forces and moments to the object's wrench.
3. **Find admissible contact loads.** Balance the external wrench within friction and compression limits.
4. **Check robot effort.** Can the fingers and joints supply the required forces and torques?

So far, we knew where the fingers would touch. **How can a depth camera help us propose those contacts?**

A depth view samples visible surfaces

visible surface samples



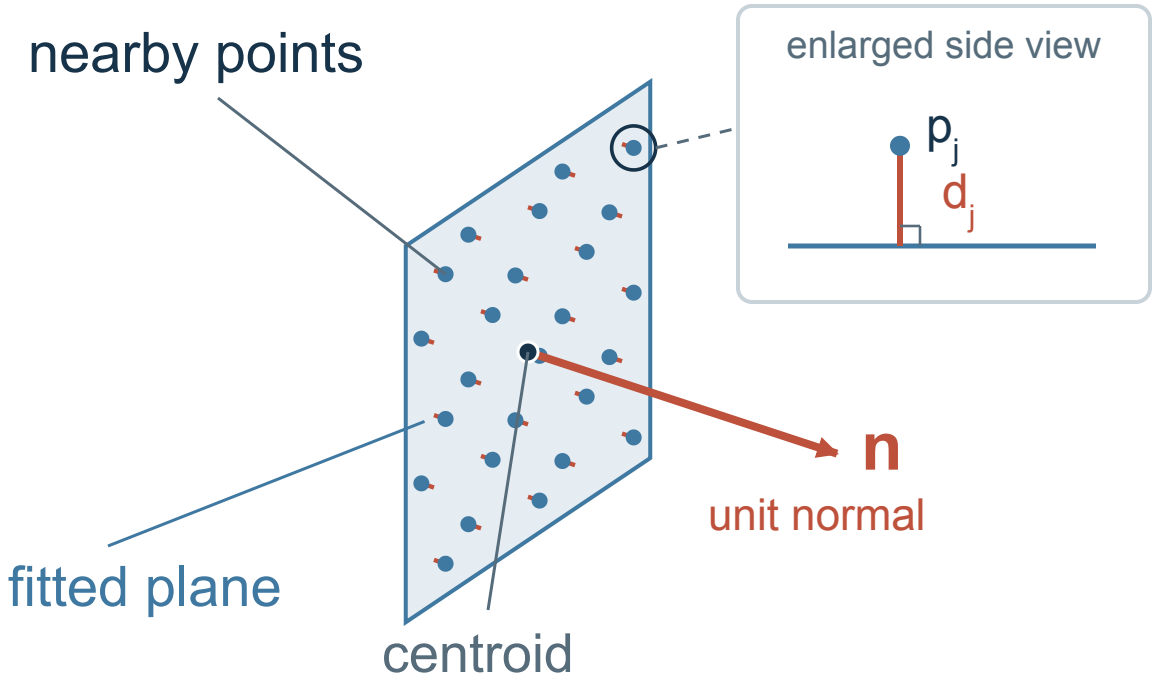
Each blue dot represents a 3D position measured on a visible face. Together these samples form a **point cloud**.

Use the known camera pose to express each point in the world frame.

$$ {}^{W}\mathbf{p}^{P_i} = {}^{W}\mathbf{R}^{C}\,{}^{C}\mathbf{p}^{P_i} + {}^{W}\mathbf{p}^{C}. $$

Camera C · world W · world position $\mathbf{p}_i$ (m)

Fit a local plane to estimate surface orientation



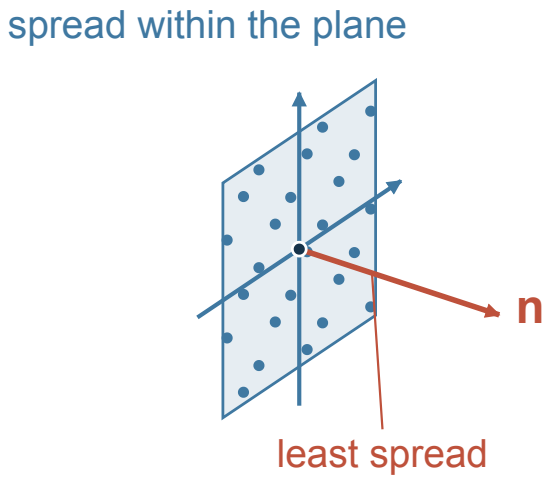
At P_i, which way does the surface face?

Fit a plane through the neighbors' mean position $\bar{\mathbf{p}}_i$. For a trial unit normal $\mathbf{n}$, neighbor j lies a distance d_j from the plane.

$$d_j = \left|\mathbf{n}^{\mathsf{T}}(\mathbf{p}_j - \bar{\mathbf{p}}_i)\right|.$$

Choose the normal that **minimizes** $\sum_j d_j^2$ over the neighborhood.

The normal is the direction of least spread



$\mathcal{N}_i$ contains the indices of the N_i neighbors.

1. Center the neighboring points.

$$\bar{\mathbf{p}}_i = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \mathbf{p}_j, \qquad \mathbf{u}_j = \mathbf{p}_j - \bar{\mathbf{p}}_i.$$

2. Compute their covariance.

$$\mathbf{S}_i = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \mathbf{u}_j \mathbf{u}_j^\mathsf{T}.$$

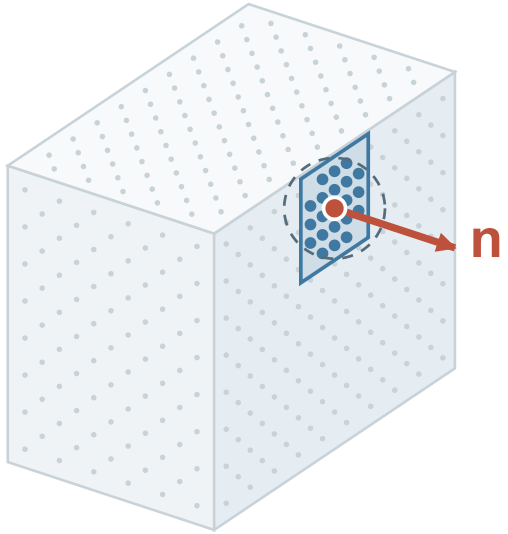
3. Find the best-fitting unit normal.

$$\hat{\mathbf{n}}_i \in \arg\min_{\|\mathbf{n}\|=1} \underbrace{\mathbf{n}^\mathsf{T} \mathbf{S}_i \mathbf{n}}_{\frac{1}{N_i} \sum_{j \in \mathcal{N}_i} (\mathbf{n}^\mathsf{T} \mathbf{u}_j)^2}$$

The fitted normal is a unit eigenvector of $\mathbf{S}_i$ for its **smallest eigenvalue**.

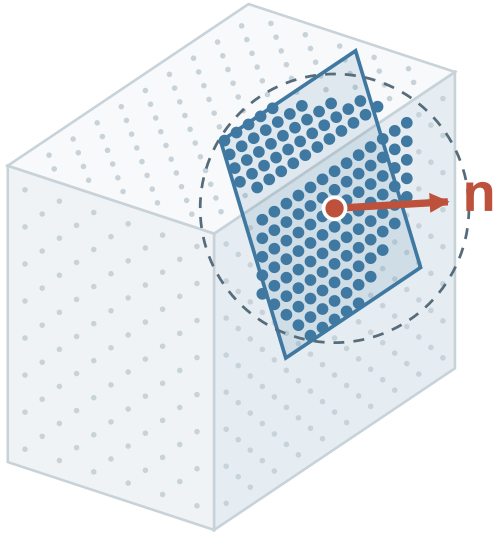
A larger neighborhood can mix two faces

radius 12 mm



one face

radius 32 mm

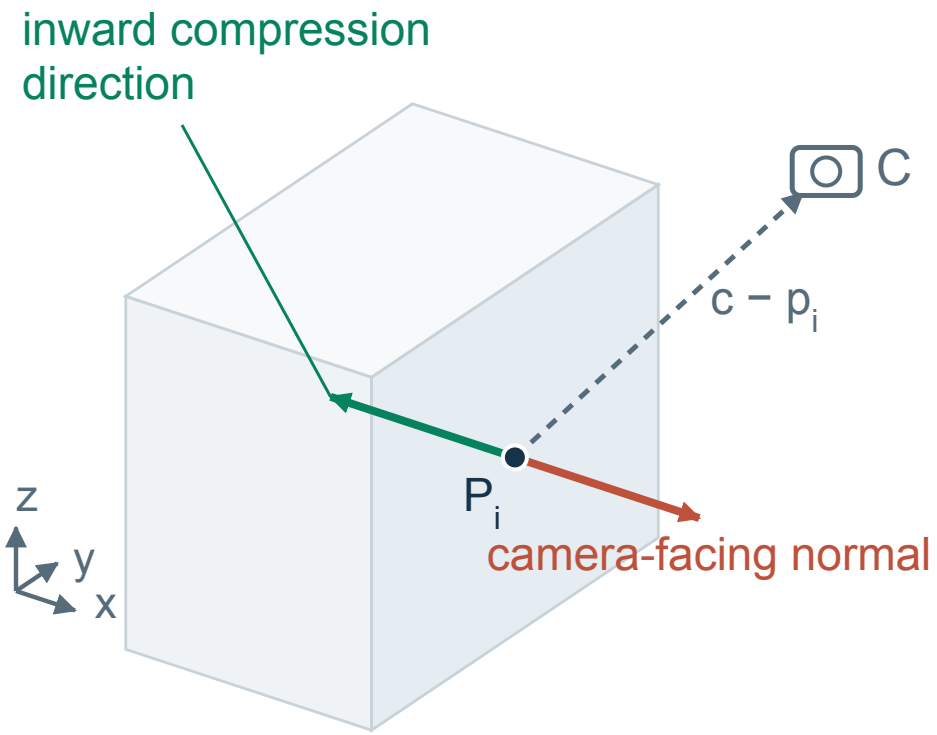


two faces in the fit

At the same side-face point, including top-face samples **tilts the estimate away from the side-face normal.**

The radius selects samples for the fit.

Use the source camera to choose the normal's sign



Both $\mathbf{n}_i$ and $-\mathbf{n}_i$ fit the plane equally well.

From P_i, trace $\mathbf{c} - \mathbf{p}_i$ toward the camera. Choose the sign of $\mathbf{n}_{\text{obs},i}$ that points **toward the camera side of the plane**.

$$\mathbf{n}_{\text{obs},i}^{\mathsf{T}}(\mathbf{c} - \mathbf{p}_i) \geq 0.$$

$\mathbf{c}$ is the source camera's position in the world frame.

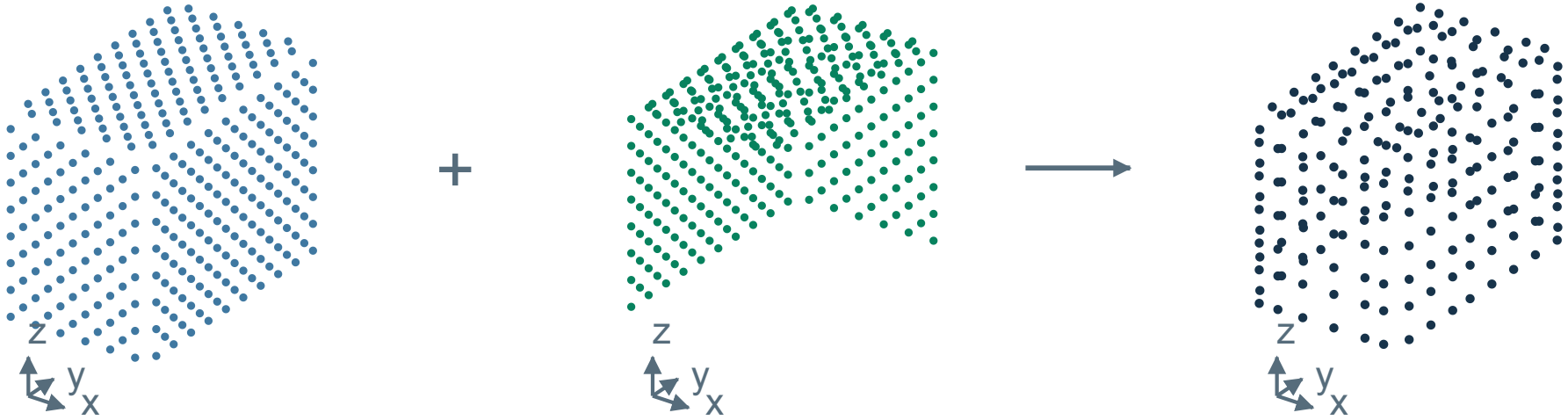
At a right angle, either sign gives zero.

Orient each view’s normals before merging the clouds

View A

View B

Combined + reduced

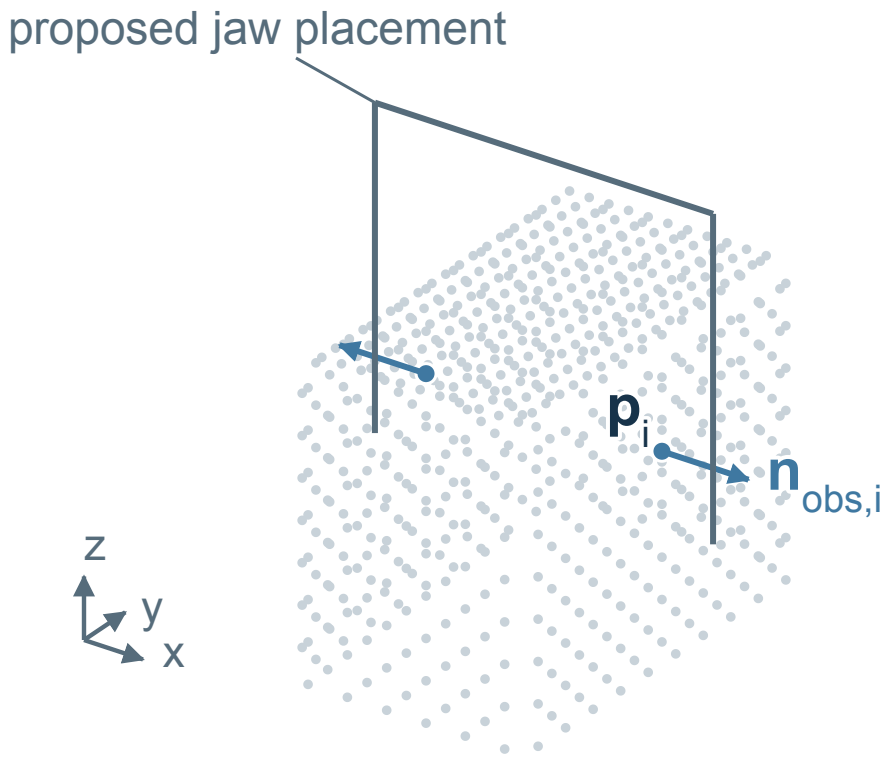


First, work in each view separately. Fit a local plane around each sample. Of its two normal directions, keep the one pointing toward the camera that saw the sample.

Then merge the oriented clouds. Keep one point per occupied voxel (a small 3D grid cell).

Demo · two camera views ↗

Use surface geometry to propose finger placements



Where could a finger touch?

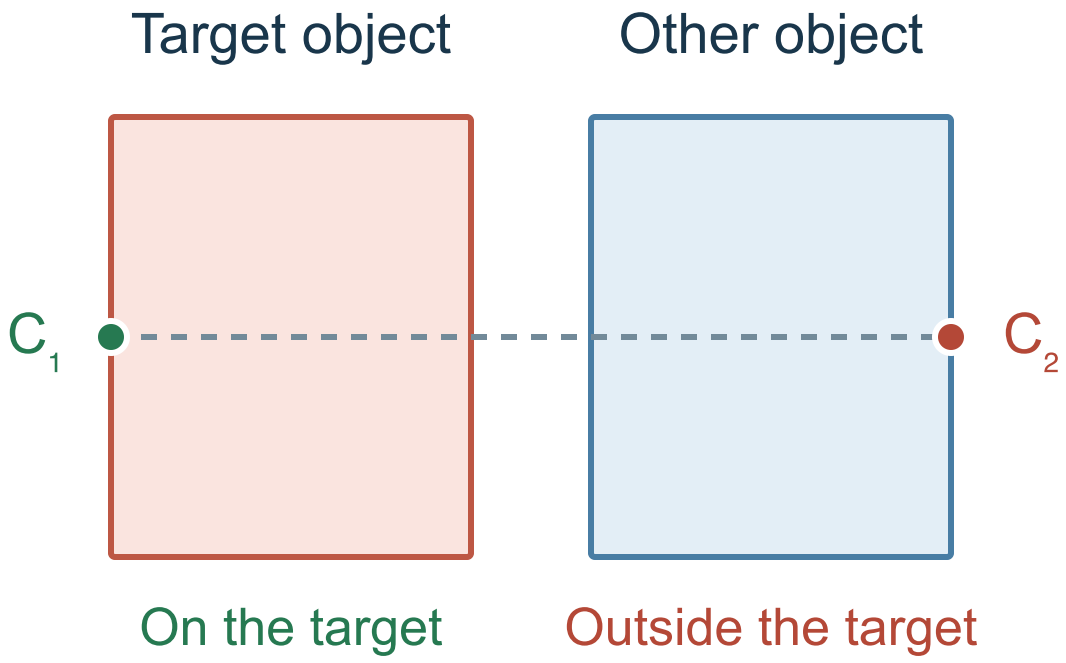
The sample position $\mathbf{p}_i$ gives a possible contact location.

Which way could it press?

On this exterior face, $\mathbf{n}_{\text{obs},i}$ points outward. Pure normal compression acts along $-\mathbf{n}_{\text{obs},i}$.

In a cluttered scene, which samples belong to the object we want to grasp?

Which surface points belong to the target?



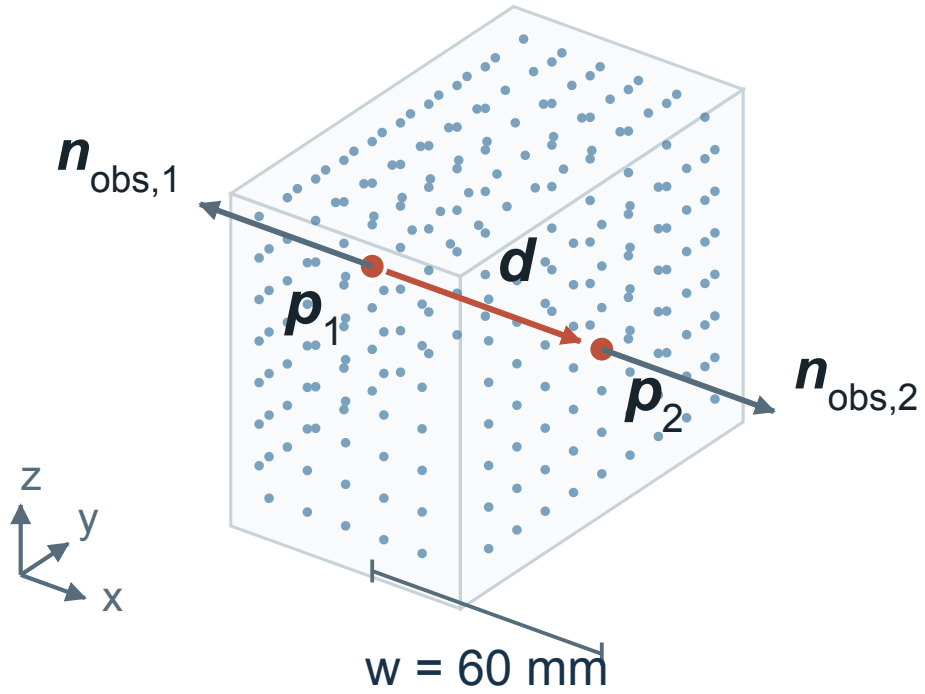
Schematic

Confirm the red box in **each camera image**.

Its mask selects pixels. Calibrated depth gives the 3D positions of their observed surface points.

Keep a pair only if both points belong to the target.

Two target points define a candidate pinch



Choose one **target sample $\mathbf{p}_1$**. Search for partners $\mathbf{p}_2$ within the gripper's contact-width range.

Separation at contact

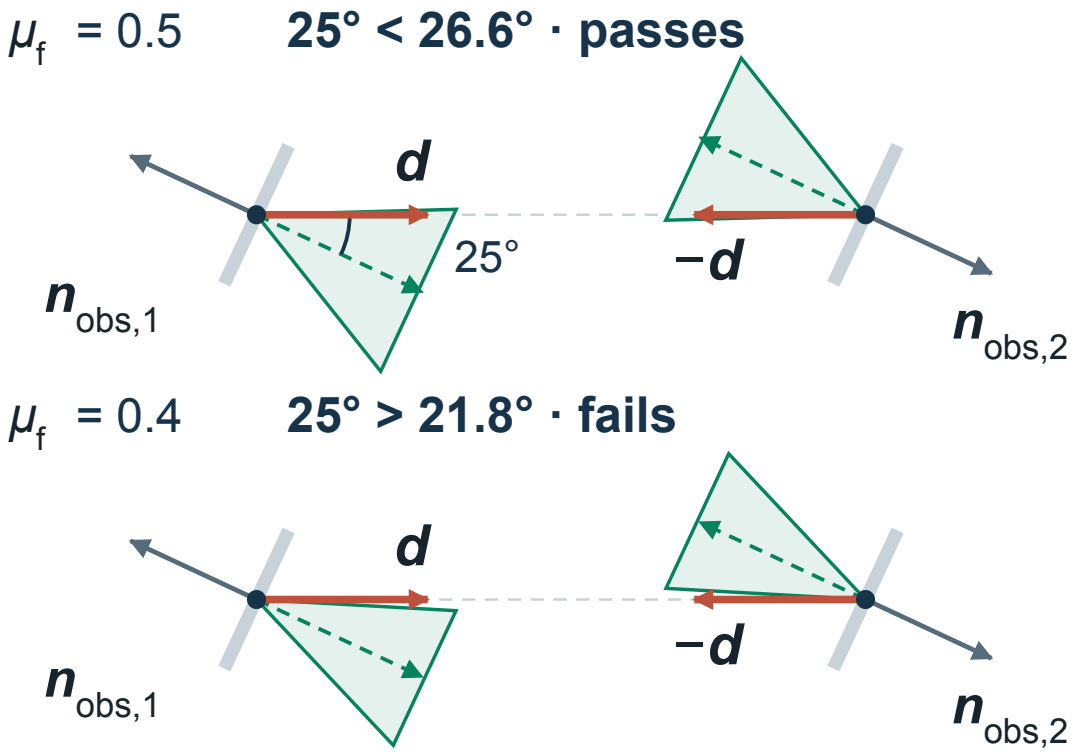
$$w = \|\mathbf{p}_2 - \mathbf{p}_1\|$$

Unit direction from 1 to 2

$$\mathbf{d} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{w}$$

For each pair with $w_{\min} \leq w \leq w_{\max}$, test the joining line at both surfaces.

Can both contacts support the proposed squeeze?



The observed normals point outward. Each proposed push must lie inside or on its **inward friction cone**.

Contact	Push · inward normal	Keep if
1	$(+\mathbf{d})^{\mathsf{T}}(-\mathbf{n}_{\text{obs},1})$	$\geq \cos\alpha_{\text{f}}$
2	$(-\mathbf{d})^{\mathsf{T}}(-\mathbf{n}_{\text{obs},2})$	$\geq \cos\alpha_{\text{f}}$

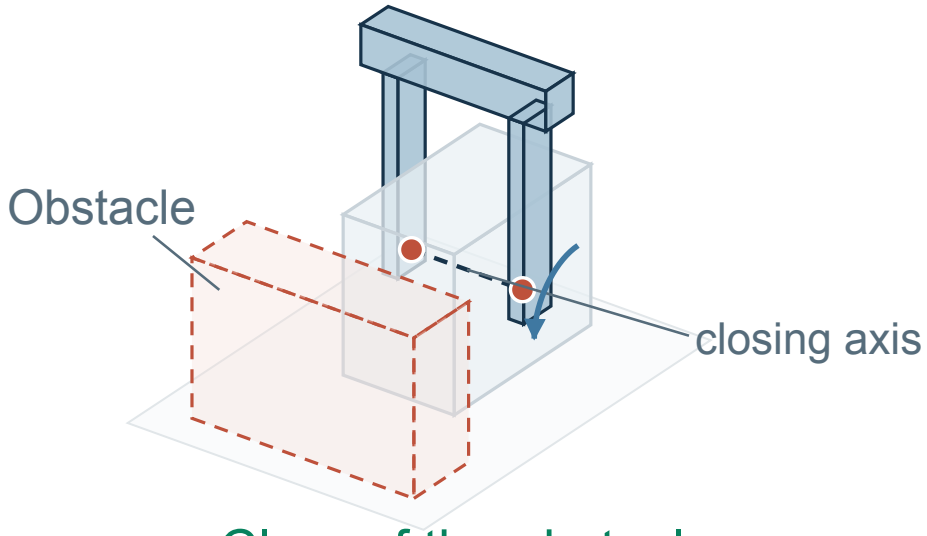
Cone half-angle $\alpha_{\text{f}} = \arctan\mu_{\text{f}}$ for assumed friction coefficient μ_{f}.

If both pushes fit, keep the squeeze candidate. Can it balance the task's forces and moments?

Where can the palm go if the contacts stay fixed?

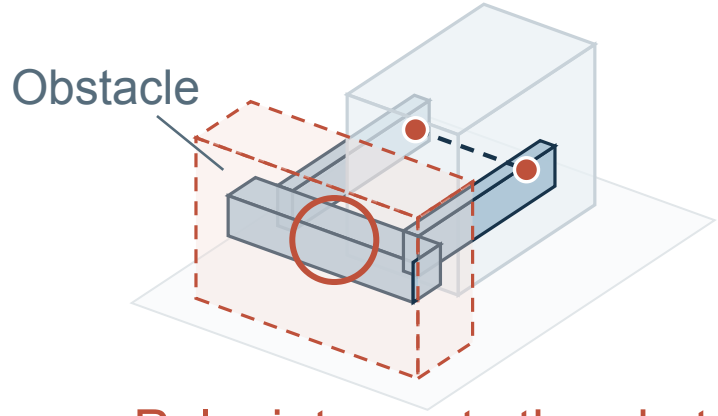
Return to S49's **60 mm box pair**. The two points fix the midpoint and closing axis. Try rotating the gripper about that axis.

Palm above the box



Clear of the obstacle

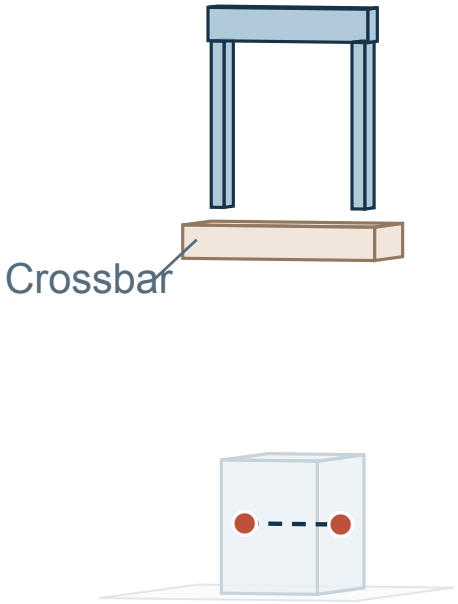
Palm rolled 90° about the closing axis



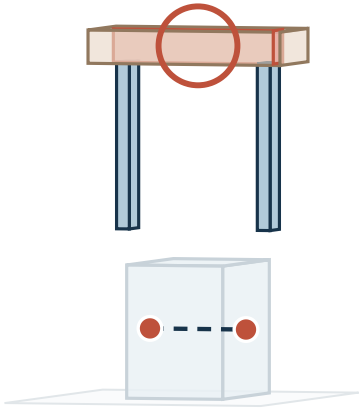
Palm intersects the obstacle

A clear final pose can have a blocked approach

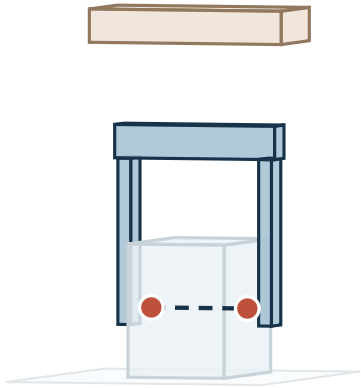
Start above the bar



Palm meets the bar



Final pose below the bar



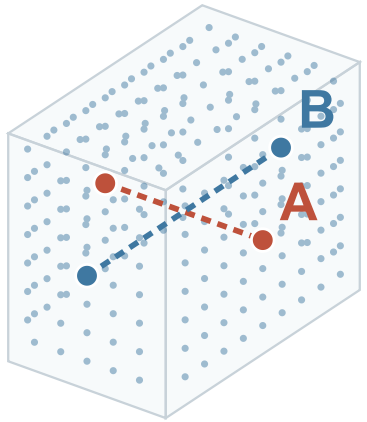
Reject this straight approach. Try another path or gripper roll for the same pair.

Check the open fingers, palm and robot arm along the path.

Which candidate pair should we try first?

$$s_{\text{pair}} = \underbrace{\frac{|\mathbf{d}^{\mathsf{T}}\mathbf{n}_{\text{obs},1}| + |\mathbf{d}^{\mathsf{T}}\mathbf{n}_{\text{obs},2}|}{2}}_{\text{alignment reward}} - \underbrace{0.2\frac{w}{w_{\max}}}_{\text{width penalty}}$$

A · 60 mm



B · 100 mm

Both ideal pairs pass S50 and have alignment reward 1. Here $w_{\max} = 180$ mm.

Pair	Score	Priority
A	$1 - 0.2(60/180) \approx 0.933$	First
B	$1 - 0.2(100/180) \approx 0.889$	Later

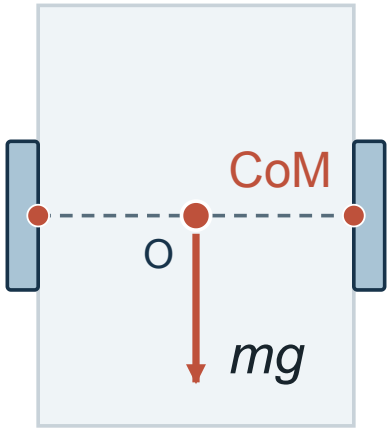
The weight 0.2 expresses a preference for smaller contact separation.

Will the highest-ranked pair hold this load?

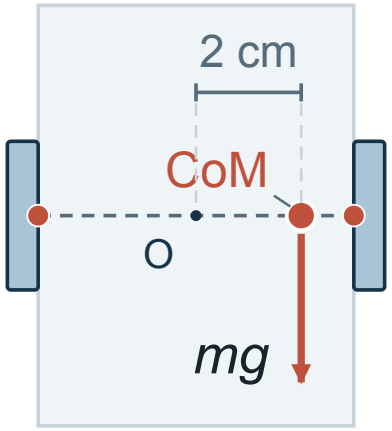
Pair A from S53 · 60 mm · geometric score $s_{\text{pair}} \approx 0.933$ in both cases

Planar point-contact pinch · mass 1 kg · friction coefficient 0.5 · compression cap 15 N per finger

Centered mass



Mass shifted right



Minimum squeeze per finger = larger required upward contact force divided by μ_{f}

$$(4.905\ \text{N})/0.5 = 9.81\ \text{N}$$

Within the 15 N cap

$$(8.175\ \text{N})/0.5 = 16.35\ \text{N}$$

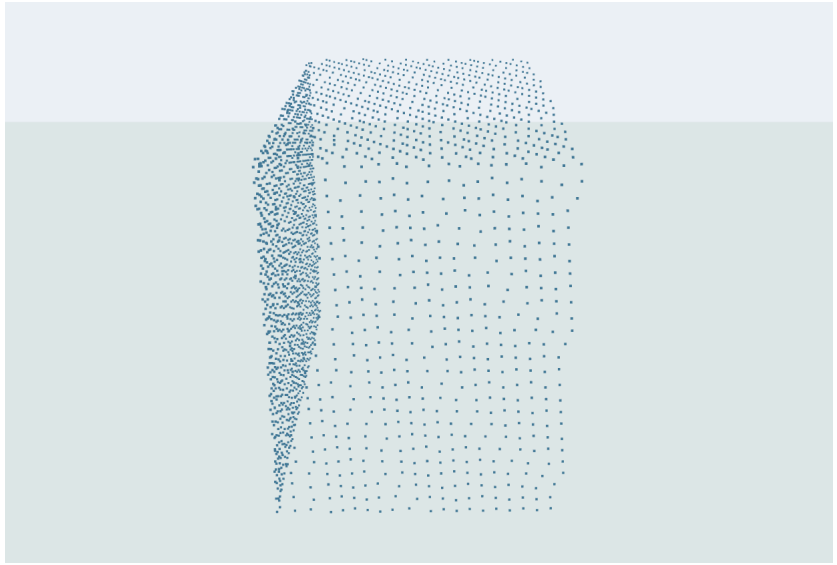
Exceeds the 15 N cap

A score is not a load check

Classroom demo (req. Drake install)

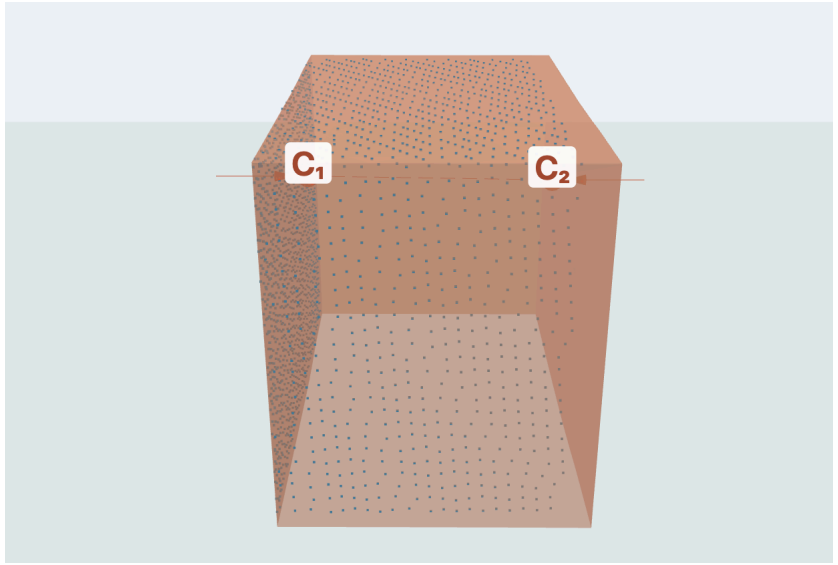
What if the camera never saw a second contact?

Observed samples



The sampler finds no pair on these samples.

Observed samples + inferred surface



Both endpoints come from the inferred surface.

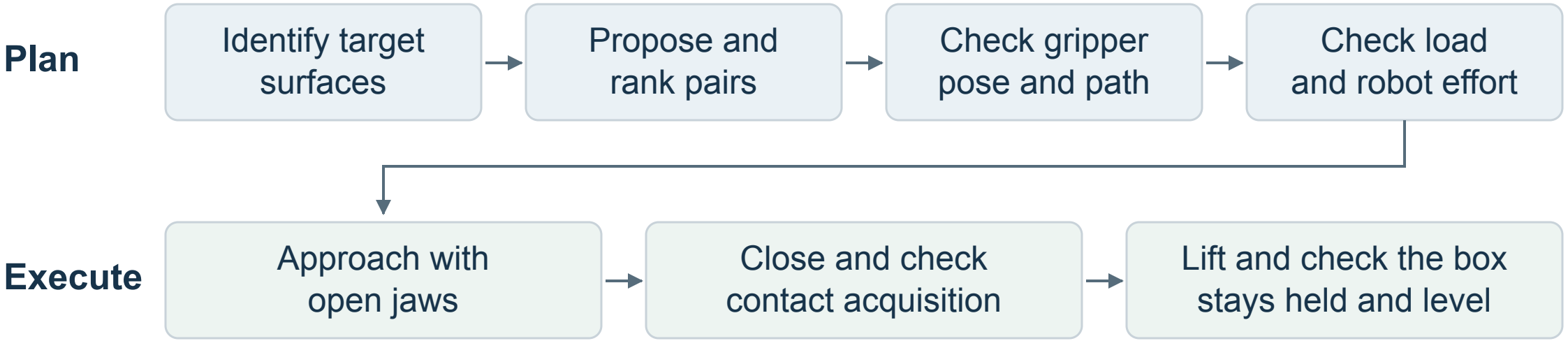
C_1 agrees with visible depth. C_2 lies behind the visible surface along its camera ray.

Elsewhere, parts of the inferred surface conflict with Camera A's measured depth.

Demo · inspect hidden contact ↗

From a proposed pair to a tested lift

Task · Lift the selected box and keep it level.



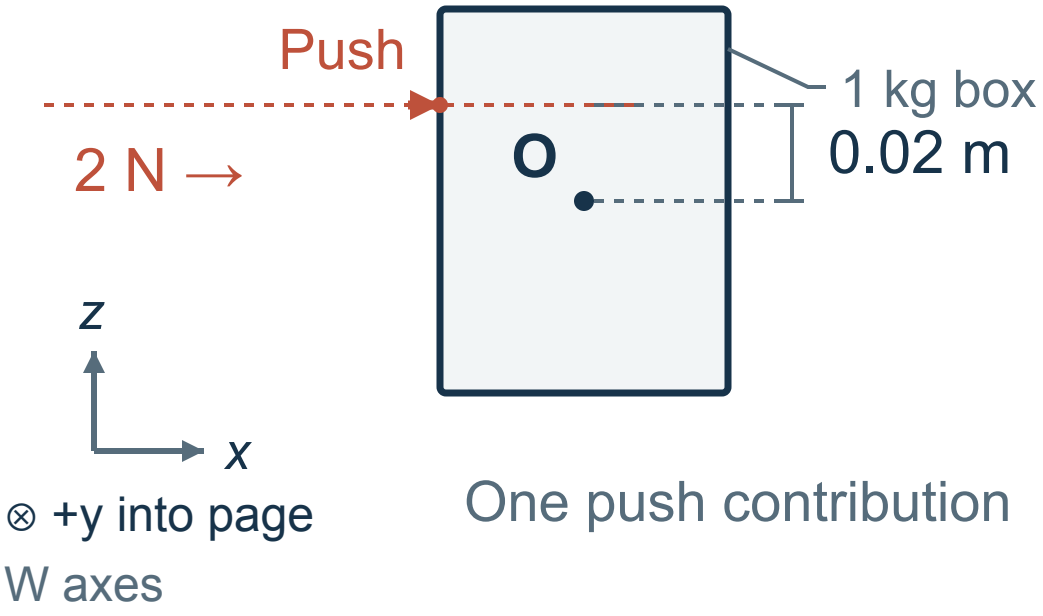
Blocked approach? Try another roll or path. **Uncertain hidden contact?** Seek another view.

End of the main lecture

The remaining slides are backup material for questions and review.

Worked examples · Symbol tables · Sources

Worked example: the component cross product



One push contribution

Position from O × force:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} yf_z - zf_y \\ zf_x - xf_z \\ xf_y - yf_x \end{bmatrix}$$

In this plane: $y = 0$, $f_y = 0$.

$$\tau_y = zf_x - xf_z$$

x, y, z: position from O.
f_x, f_y, f_z: force components.

$$\tau_y = +0.04\ \mathrm{N\,m}$$

Return to explanation ↩

One force, two reporting points

General case: one force $\mathbf{f}$ applied at S.

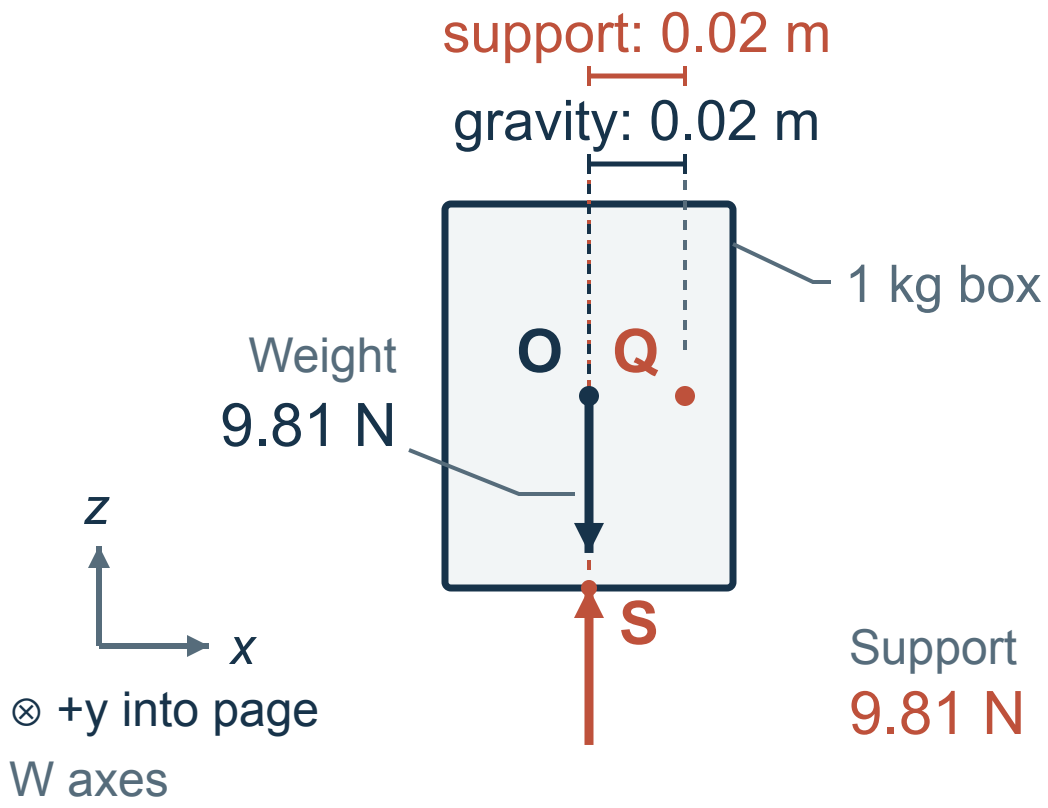
$$\boldsymbol{\tau}_Q = {}^{Q}\mathbf{p}_W^S \times \mathbf{f} = \boldsymbol{\tau}_O + {}^{Q}\mathbf{p}_W^O \times \mathbf{f}$$

$$\boldsymbol{\tau}_O = {}^{O}\mathbf{p}_W^S \times \mathbf{f}$$

${}^{Q}\mathbf{p}_W^S$: Q to S; ${}^{Q}\mathbf{p}_W^O$: Q to O; ${}^{O}\mathbf{p}_W^S$: O to S. All vectors use world axes W.

Special case: if the force's line of action passes through O, then $\boldsymbol{\tau}_O = \mathbf{0}$.

The two moments cancel about Q



Moments about Q · N m

$$\text{support:} \quad +0.1962$$

$$\text{gravity:} \quad -0.1962$$

$$\tau_y = 0 \text{ N m}$$

Force balance is unchanged.

A 2 N push remains supported



$$f_n \approx 9.810\ \mathrm{N} \qquad f_t \approx -2.000\ \mathrm{N}$$

Recorded Drake state · $h = 0.06\ \mathrm{m}$ · pale shape: initial pose
Support and pressure view

Return to explanation ↩

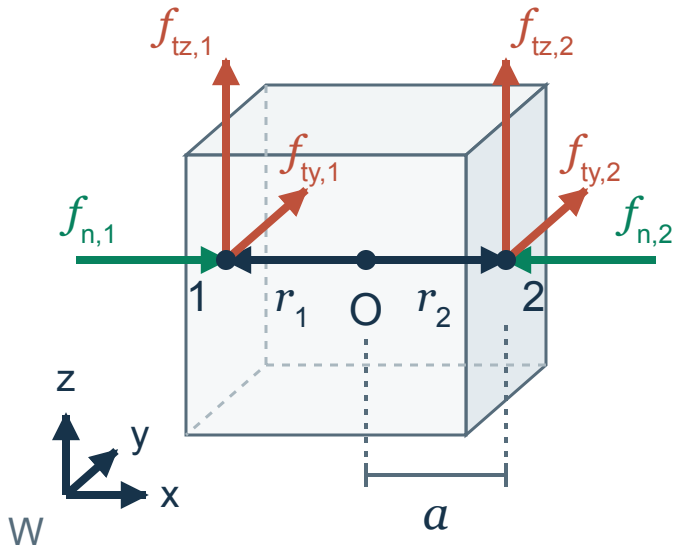
Simulation: supported load

Two point contacts: the 3D grasp matrix

$\mathbf{r}_1 = [-a, 0, 0]^\mathsf{T}$, $\mathbf{r}_2 = [a, 0, 0]^\mathsf{T}$, $a = 0.03\,\mathrm{m}$: half-width along x · moments about O, world axes W.

$\mathbf{G}_{\text{grasp}} \in \mathbb{R}^{6\times 6}$ · columns: force inputs · rows: total wrench

	$f_{n,1}$	$f_{t_y,1}$	$f_{t_z,1}$	$f_{n,2}$	$f_{t_y,2}$	$f_{t_z,2}$
τ_x	0	0	0	0	0	0
τ_y	0	0	a	0	0	$-a$
τ_z	0	$-a$	0	0	a	0
f_x	1	0	0	-1	0	0
f_y	0	1	0	0	1	0
f_z	0	0	1	0	0	1



One column of the 3D grasp matrix:

$$\begin{bmatrix} \mathbf{r}_i \times \mathbf{d}_i \\ \mathbf{d}_i \end{bmatrix}$$

Wrench per unit force along $\mathbf{d}_i$ at contact i.

$\mathbf{r}_i$: vector from O to contact i.

$\mathbf{d}_i$: unit force direction. Normals: $+x$ at contact 1, $-x$ at contact 2.

Tangents: $+y$ or $+z$ at either contact.

Return to explanation ↩

Choose forces that satisfy all three constraints

Fixed geometry, planar friction → **linear program (LP)**

Contacts at $x = \pm a$, $z = 0$; x_{com}: CoM offset from O along x.
a, m, g, x_{com} and μ_{f} are fixed.

Compact formulation

$$\min_{\mathbf{f}_{\text{c}},\, s} \; s$$

Balance

$$\mathbf{G}_{\text{grasp}}\mathbf{f}_{\text{c}} + \mathbf{F}_g = \mathbf{0}$$

Friction

$$|f_{t,i}| \le \mu_{\text{f}} f_{n,i}$$

Compression

$$0 \le f_{n,i} \le s, \quad f_{n,i} \le 15\ \text{N}$$

Two-contact example

$$\min_{f_{n,1},\, f_{t,1},\, f_{n,2},\, f_{t,2},\, s} \; s$$

$$\begin{aligned}
a f_{t,1} - a f_{t,2} + m g x_{\text{com}} &= 0 \\
f_{n,1} - f_{n,2} &= 0 \\
f_{t,1} + f_{t,2} - m g &= 0 \\
f_{t,1} \le \mu_{\text{f}} f_{n,1}, \qquad -f_{t,1} &\le \mu_{\text{f}} f_{n,1} \\
f_{t,2} \le \mu_{\text{f}} f_{n,2}, \qquad -f_{t,2} &\le \mu_{\text{f}} f_{n,2} \\
0 \le f_{n,1} \le s, \qquad f_{n,1} &\le 15\ \text{N} \\
0 \le f_{n,2} \le s, \qquad f_{n,2} &\le 15\ \text{N}
\end{aligned}$$

s: compression bound (N); $i = 1, 2$. **All constraints hold simultaneously.**

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The contact-force linear program

Open live demo ↗

Compute the Jacobian for the one-link arm

$L = 0.30$ m; q: counterclockwise angle from horizontal, about $-y$.

Endpoint position and Jacobian

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} L\cos q \\ L\sin q \end{bmatrix}$$

$$\mathbf{J}_c(q) = \frac{\partial(x, z)}{\partial q} = \begin{bmatrix} -L\sin q \\ L\cos q \end{bmatrix}$$

$\mathbf{J}_c \in \mathbb{R}^{2\times 1}$: one joint velocity maps to two endpoint velocity components, $\dot{x}$ and $\dot{z}$.

The two arm configurations

q	$\mathbf{J}_c(q)$	$\tau_{\text{joint,load}}$
0°	$\begin{bmatrix} 0 \\ 0.30 \end{bmatrix}$	$2.943\ \text{N m}$
60°	$\begin{bmatrix} -0.2598 \\ 0.15 \end{bmatrix}$	$1.4715\ \text{N m}$

$$\mathbf{f}_{\text{tip}} = \begin{bmatrix} 0 \\ 9.81 \end{bmatrix}\text{N}, \qquad \tau_{\text{joint,load}} = \mathbf{J}_c^{\mathsf{T}}\mathbf{f}_{\text{tip}} = (-L\sin q)(0) + (L\cos q)(9.81)$$

Upward force × horizontal lever arm. Joint effort compensates the force reaction; robot self-weight excluded.

Worked example: matching a wrench to a Jacobian

$$\mathbf{V}_P = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_P \end{bmatrix} = \mathbf{J}_P \dot{\mathbf{q}}, \qquad \mathbf{F}_P = \begin{bmatrix} \boldsymbol{\tau}_P \\ \mathbf{f} \end{bmatrix}$$

$$\mathbf{F}_P^{\mathsf{T}} \mathbf{V}_P = \left(\mathbf{J}_P^{\mathsf{T}} \mathbf{F}_P\right)^{\mathsf{T}} \dot{\mathbf{q}}$$

$$\boldsymbol{\tau}_{\text{joint,load}} = \mathbf{J}_P^{\mathsf{T}} \mathbf{F}_P$$

P: robot tool/load interface. Match moment reference point, expressed-in frame, and angular-first order.
$\mathbf{J}_P \in \mathbb{R}^{6\times n}$; n: number of joints.

$\mathbf{F}_P$ is exerted **by the robot on the load**. The effort above compensates its reaction; robot self-weight is separate.

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A changed score can mean two different things

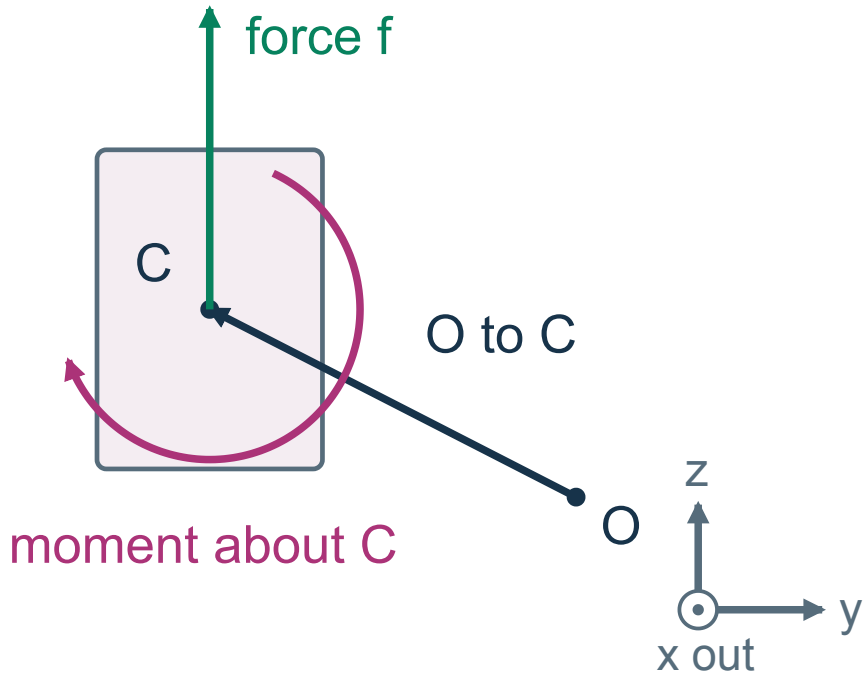
Same box and +1 cm load; $F_{\mathrm{ref}} = 9.81\ \mathrm{N}$ throughout.

Normal caps, left / right	ℓ_{ref}	$\epsilon_{\mathrm{grasp}}$	**Hold this load?**
15 N / 15 N	0.03 m	0.883	Yes
15 N / **12 N**	0.03 m	0.706	**No**
15 N / 15 N	**0.06 m**	0.624	Yes

A lower force cap changes **physical capacity**.
A different reference length changes **how we measure it**.

Return to explanation ↩

A patch contributes a force and a moment



$$\boldsymbol{\tau}_O = {}^{O}\mathbf{p}_W^C \times \mathbf{f} + \boldsymbol{\tau}_C$$

Lever-arm moment: position from O to C crossed with the resultant force.

Moment about C: the distributed contact's remaining moment.

C: contact-surface centroid used by the demo. All vectors expressed in world axes W.

Worked example: recover a tilted plane normal

Four centered samples, in mm.
Columns are points.

$$\mathbf{D} = \begin{bmatrix} -10 & -10 & 10 & 10 \\ -10 & -10 & 10 & 10 \\ -10 & 10 & -10 & 10 \end{bmatrix}.$$

Every point satisfies $x = y$.

The plane's perpendicular direction must have opposite x and y components.

Average the centered outer products:

$$\mathbf{S} = \tfrac{1}{4}\mathbf{D}\mathbf{D}^{\mathsf{T}} = \begin{bmatrix} 100 & 100 & 0 \\ 100 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{ mm}^2.$$

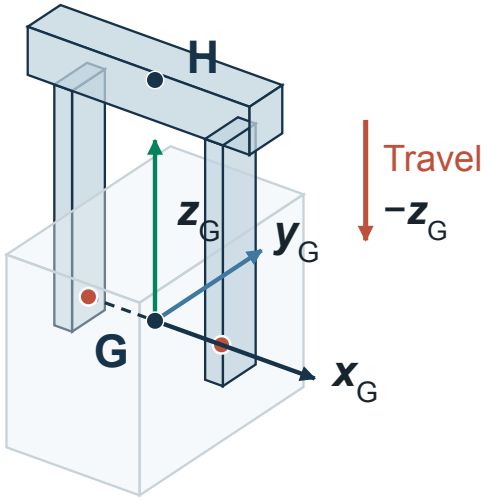
The smallest eigenvalue is zero:

$$\mathbf{S}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{0}.$$

So $\mathbf{n} = \pm(1, -1, 0)^{\mathsf{T}}/\sqrt{2}$.

Original numerical example · MIT: local plane fitting

Worked example: complete a gripper pose



G: contact-center frame.
H: chosen palm/tool reference.
Approach travel is along $-\mathbf{u}$.

Place the origin at the contact midpoint:

$$\mathbf{p}_{\text{mid}} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}.$$

Choose a unit pregrasp-side direction $\mathbf{u} \perp \mathbf{d}$:

$${}^{W}\mathbf{R}^{G} = \begin{bmatrix} \mathbf{d} & \mathbf{u} \times \mathbf{d} & \mathbf{u} \end{bmatrix}.$$

Apply the fixed tool offset $\mathbf{r}_{GH} = {}^{G}\mathbf{p}_{G}^{H}$:

$${}^{W}\mathbf{p}_{W}^{H} = \mathbf{p}_{\text{mid}} + {}^{W}\mathbf{R}^{G}\mathbf{r}_{GH}.$$

Here $\mathbf{d} = (1, 0, 0)$, $\mathbf{u} = (0, 0, 1)$, so ${}^{W}\mathbf{R}^{G} = \mathbf{I}$.

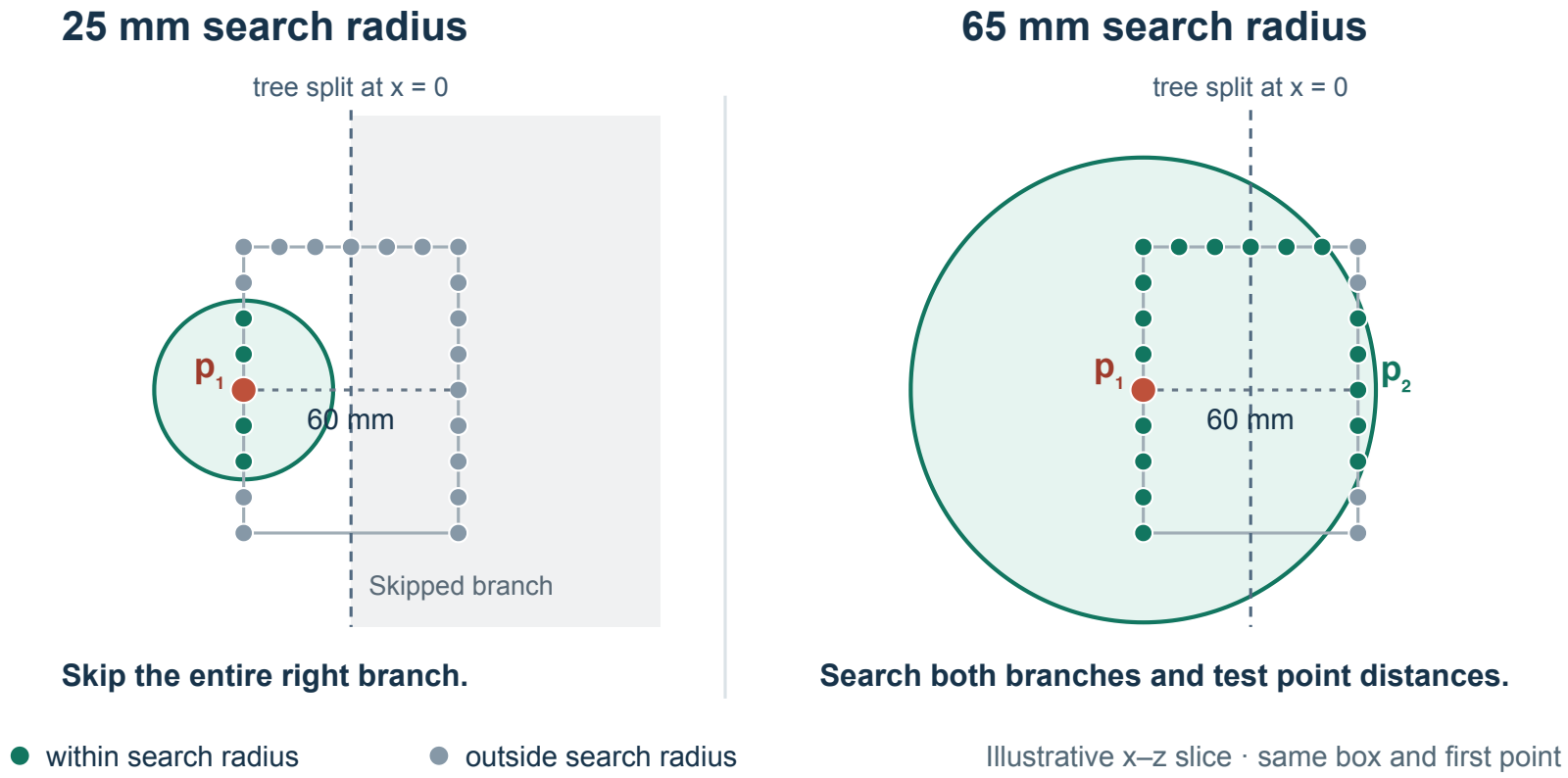
$\mathbf{p}_{\text{mid}} = (0, 0, 0.04)\ \text{m}$, $\mathbf{r}_{GH} = (0, 0, 0.10)\ \text{m}$.

Thus ${}^{W}\mathbf{p}_{W}^{H} = (0, 0, 0.14)\ \text{m}$.

Return to placement ↩

How a k-d tree skips distant samples

Build the tree once. Each split uses one coordinate, cycling through x, y and z.



Course preset: the 180 mm radius covers this entire box. The width and friction-cone tests still apply.

Force and wrench symbols

Symbol	Meaning and units
W, B	World frame; box body
O, Q; S, S_h	Reporting points; force application points
x, z; d	Position relative to the reporting point; perpendicular lever arm (m)
f_{sup}; $\mathbf{f}$	Upward support magnitude; force vector (N)
$\boldsymbol{\tau}$, τ_y	Moment vector; planar component (N m)
$\mathbf{F}^{B_O}_{\mathrm{net},W}$	Net wrench on B, about O, in W; moment first
${}^{Q}\mathbf{p}^{O}_{W}$, ${}^{Q}\mathbf{p}^{S}_{W}$	Vectors from Q to O or S, expressed in W (m)

Contact and support symbols

Symbol	Meaning and units
f_n, f_t	Compressive normal force; signed tangential force (N)
μ_{f}	Friction coefficient (dimensionless)
f_{sq}, f_p	Compression per finger; horizontal push magnitude (N)
a, h	Support half-length; push height above the table (m)
θ, c	Ramp angle; CoM height perpendicular to the bottom face (m)
T, x_{cp}	Support midpoint; normal-resultant position relative to T (m)
$\tau_{y,T}^{\mathrm{ct}}$	Contact moment about T, along y (N m)
f_{slide}, f_{tip}	Candidate static sliding and tipping thresholds (N)

Sources and example status

- Russ Tedrake, *Robotic Manipulation*, Spatial force and spatial-force algebra: wrench definitions and reporting-point shifts.
- Tomás Lozano-Pérez, Fall 2026 lecture: wrench and object-support motivation.
- Drake, Modeling Compliant Contact: distributed contact forces and torsional friction.
- Lynch and Park, *Modern Robotics*: §12.2.1 Friction and §12.2.3 Force closure. Spatial friction, directional closure, and the two-point limitation.

Original course examples: analytical schematics and checked Drake recordings. The contact-model traction cutaway is illustrative, not a measured field.

Force allocation and robot effort: sources

- Jeannette Bohg, CS237B Lecture 8, pp.41–50: fixed-contact equilibrium, friction and force allocation.
- Boyd and Wegbreit, *Fast Computation of Optimal Contact Forces*, 2007: conic force optimization.
- Lynch and Park, *Modern Robotics*, §5.2: Statics of Open Chains: Jacobian transpose and the power argument.

The force-allocation examples are original and analytical. Their LP minimizes normal compression.

Grasp quality and selection: sources

- Ferrari and Canny, *Planning Optimal Grasps*, §§4.1–4.4, 1992: force budgets, wrench scaling and weakest-direction quality.
- Jeannette Bohg, CS237B Lecture 8, pp.33–36, 2025: geometric interpretation of grasp quality.
- Course antipodal-grasp demo: signed pair screen and complete pair score.
- Russ Tedrake, Bin Picking: candidate generation, collision rejection and a different geometric cost.

The grasp stress test uses declared force caps and wrench scaling. The antipodal pair score is a geometric preference. Its pose and load comparisons are original course constructions.